Lecture 4

Curves

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“A smile is a curve that sets everything straight...”

Phyllis Diller
(American comedienne and actress, born 1917)
Outline

- Introduction
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- Curves
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Introduction

We will discuss some of the existing curves associated with their interpolation techniques. We will start with few of the important properties which we desire for curves e.g. continuity and affine invariance. We will introduce transformations for manipulating curves and then proceed to discuss some of the most popular curves used in computer graphics. But let’s first introduce the notion of parametric curves which will be later extended to define parametric surfaces.
Basic Building Block – A Point

- Let’s first start with the basic building block of the Euclidean space, a **point**, which can be defined as follows:

**Definition 1**: Real Euclidean $d$-space is given by $\mathbb{R}^d = \{x = (x_1, x_2, ..., x_d) | x_i \in \mathbb{R}\}$ where $x$ denotes a point with $d$-coordinates.

**Example**: $\mathbb{R}^1 = \mathbb{R}$ is the real line while $\mathbb{R}^2 = \{x = (x, y)\}$ is the standard Euclidean plane and $\mathbb{R}^3 = \{x = (x, y, z)\}$ is the three dimensional space.
A line segment can be defined by its ending points.

Many ways to define the equation of a line.

Think about it as a road, we will end up with the parametric form of the line.

Think of it as if you started at $x_0$ then walk $t$ of the distance between $x_0$ and $x_1$, hence your current position will be defined as follows;

$$x(t) = x_0 + t(x_1 - x_0) = x_0 + tx_1 - tx_0 = (1-t)x_0 + tx_1 \quad (1)$$

Hence, what we have done now is representing any point on the line connecting $x_0$ and $x_1$ by a weighted average of the two ending points, this average is parameterized by one parameter $t$. 
We can re-write (1) as:
\[ x(t) = f_0(t)x_0 + f_1(t)x_1 \]

(2)

Where \( f_0(t) = 1 - t \) and \( f_1(t) = t \).

Since we are representing a line, the degree of \( f_k(t) \) for \( k = 0,1 \) is one (i.e. linear functions), hence we only need two points to represent a line segment.

Eq(2) can be thought of as representing a line segment with control points \( x_0 \) and \( x_1 \) being interpolated with basis functions \( f_k(t) \). This equation is usually referred to as the parametric equation of a line, let’s now generalize this notion to curves and later to surfaces, where curves are represented by non-linear basis functions and hence we need more control points.
Affine Transformations

• Basically, we would like to be able to transform a curve without manually calculating the each individual point in the new curve or surface. Affine transforms will allow us to do this. But first what is a transform?

**Definition 2:** A transform/warp on \( \mathbb{R}^d \) is any mapping \( W: \mathbb{R}^d \rightarrow \mathbb{R}^d \). That is, each point \( \mathbf{x} \in \mathbb{R}^d \) is mapped to exactly one point \( W(\mathbf{x}) \) also in \( \mathbb{R}^d \).

**Definition 3:** Let \( W: \mathbb{R}^d \rightarrow \mathbb{R}^d \) be a transform. \( W \) is said to be a linear transform/warp if and only if:

(a) For all \( \alpha \in \mathbb{R} \) and all \( \mathbf{x} \in \mathbb{R}^d \) we have \( W(\alpha \mathbf{x}) = \alpha W(\mathbf{x}) \).

(b) For all \( \mathbf{x}, \mathbf{y} \in \mathbb{R}^d \) we have \( W(\mathbf{x} + \mathbf{y}) = W(\mathbf{x}) + W(\mathbf{y}) \).

This implies that \( W(\mathbf{0}) = \mathbf{0} \) since \( W(\mathbf{0}, \mathbf{x}) = \mathbf{0}, W(\mathbf{x}) = \mathbf{0} \). An example of a linear transform/warp is the identity transform given by \( W(\mathbf{x}) = \mathbf{x} \).
**Affine Transformations**

**Definition 4:** Let $W: \mathbb{R}^d \to \mathbb{R}^d$ be a transform $W$ is said to be a **translation** if there exists $t \in \mathbb{R}^d$ so that for all $x \in \mathbb{R}^d$ we have $W(x) = x + t$. A translation moves all vectors or points by a fixed distance in a fixed direction.

**Definition 5:** An **affine** transform is a transform that can be written as $W(x) = T(L(x))$ where $L(.)$ is a linear transform and $T(.)$ is a translation. This can also be written as $W(x) = L(x) + t$ or $W = T_t L$.

Any linear transform in $\mathbb{R}^3$ can be represented by a 3x3 matrix of the following form:

$$L = \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{pmatrix}$$
Affine Transformations - Scaling

In order to change the size of an object in $\mathbb{R}^3$, if we assume that the object is centered at the origin, then scaling is given by:

$$S = \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix}$$

To scale a point we apply the matrix $S$ to the point $\mathbf{x} \in \mathbb{R}^3$ where $\mathbf{x} = \{x, y, z\}$ to get

$$\mathbf{x}' = S \mathbf{x} = \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} sx \\ sy \\ sz \end{pmatrix}$$

This is called *uniform scaling* since we change the size of the object in all directions with the same amount, however, it is not necessary to scale evenly in all directions, in this case we can define the *non-uniform scaling* matrix as follows:

$$S = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{pmatrix}$$

Where $x, y, z$ connotes the three dimensions in a 3D Euclidean space.
Affine Transformations - Translation

Translations in $\mathbb{R}^3$ cannot be written as $3 \times 3$ matrices, however to allow a unified representation for affine transforms, we can use the homogeneous coordinate system which allows us to represent an affine transform as a matrix. A translation by $t = (t_x, t_y, t_z)^T$ is given by a $4 \times 4$ matrix defined as:

$$T_t = \begin{pmatrix}
1 & 0 & 0 & t_x \\
0 & 1 & 0 & t_y \\
0 & 0 & 1 & t_z \\
0 & 0 & 0 & 1
\end{pmatrix}$$

Linear transforms can also be represented as transforms on homogeneous coordinate systems as follows:

$$L = \begin{pmatrix}
\ell_{11} & \ell_{12} & \ell_{13} & 0 \\
\ell_{21} & \ell_{22} & \ell_{23} & 0 \\
\ell_{31} & \ell_{32} & \ell_{33} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

Affine transforms can be represented as combination of a linear transform and a translation, thus can be represented by the matrix product $W = T_t L$ in the homogeneous coordinate system.

In the same manner, translation matrices can be defined in the 2D space as $S = \begin{pmatrix}
1 & 0 & t_x \\
0 & 1 & t_y \\
0 & 0 & 1
\end{pmatrix}$
Affine Transformations - Rotation

In $\mathbb{R}^3$, the rotation matrix about the z-axis can be written as:

$$R_{z,\alpha} = \begin{pmatrix}
\cos \alpha & -\sin \alpha & 0 \\
\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{pmatrix}$$

In the same manner, the rotation matrices about x and y axes are defined as follows:

$$R_{x,\alpha} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \alpha & -\sin \alpha \\
0 & \sin \alpha & \cos \alpha
\end{pmatrix}$$

And

$$R_{y,\alpha} = \begin{pmatrix}
\cos \alpha & 0 & -\sin \alpha \\
0 & 1 & 0 \\
\sin \alpha & 0 & \cos \alpha
\end{pmatrix}$$
Curves … What are they?

Definition 6: Curves in the standard 2-dimensional Euclidean space $\mathbb{R}^2$ are called plane curves which can be described either in an explicit form, $y = f(x)$, i.e. as a function graph, or implicitly as a set of points which specify an equation $f(x, y) = 0$, this equation can be represented in a parametric form as $\mathbf{x}(t) = (x(t), y(t))^T$, however we need to place restrictions on $f$ such that the solution of $f(x, y) = 0$ do not fill the entire plane.

Definition 7: Algebraic curves are curves defined by $f$ such that $f$ is a polynomial function in two variables. Equations of the first degree define straight lines, while equations of the second degree define ellipses, parabolas or hyperbolas.

Definition 8: Curves in the standard 3-dimensional Euclidean space $\mathbb{R}^3$ are called space curves which can be defined as intersections of two surfaces defined implicitly by $f(x, y, z) = 0$ and $g(x, y, z) = 0$.

Example: the intersection of two surfaces $x^2 + y^2 + z^2 = 1$ (i.e. sphere) and $y - \frac{1}{2} = 0$ (i.e. plane) is a circle.
**Parametric Curves**

**Definition 9:** Parametric curves are curves defined in the standard 3-dimensional Euclidean space $\mathbb{R}^3$ in terms of some parameter, say $t$. The curve can then be written as $\mathbf{x}(t) = (x(t), y(t), z(t))^T$ where $x(t), y(t)$ and $z(t)$ are continuous functions defined on some interval $t \in [a, b]$ where each value of $t$ defines a point on the curve. The curve is defined to be the set of all such points.

Note that parametric curves can also be defined in the standard 2-dimensional Euclidean space $\mathbb{R}^2$, hence it can be written as $\mathbf{x}(t) = (x(t), y(t))^T$.

In most cases, it is assumed that $\mathbf{x}(t)$ can be differentiated at least twice, i.e. first and second derivates are defined for $t \in [a, b]$.

**Definition 10:** Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ be a set of point in the $d$-dimensional Euclidean space, a curve can be defined in terms of these points as:

$$C: \mathbf{x}(t) = f_1(t)\mathbf{x}_1 + f_2(t)\mathbf{x}_2 + \cdots + f_n(t)\mathbf{x}_n = \sum_{k=1}^{n} f_k(t)\mathbf{x}_k$$

Where $f_k(t)$ are continuous functions defined on the interval $t \in [0,1]$. The points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are called control points and $f_k(t)$ are called the basis functions of the curve $C$. 
Parametric Curves

- Some restrictions should be places on the continuous functions defining the curve in order not to fill the space, however there are still some parametric curves that are space filling, e.g. the Peano curve.

In 1880 the Italian logician Giuseppe Peano (1858-1932) constructed the Peano curve, a base motif fractal which uses a line segment as base. The motif is dividing the line segment in three parts, and making a square up and down the middle part. This leads to a filled square, so the curve is a space-filling curve.
Affine Invariance

Given a curve defined by $\sum_{k=1}^{n} f_k(t)x_k$ where $t \in [0,1]$, we would like to investigate what happens when an affine transform is applied to the curve.

**Definition 11:** A curve is said to be affine invariant if the affine transform/warp $W(.)$ applied to the points generated by the curve, i.e. $\mathbf{x}(t) = \sum_{k=1}^{n} f_k(t)x_k$, produces precisely the same curve as transforming the control points of the curve, i.e. $\mathbf{x}_k$, and then calculating the curve, that is:

$$W \left( \sum_{k=1}^{n} f_k(t)x_k \right) = \sum_{k=1}^{n} f_k(t)W(x_k)$$

This will be satisfied if the basis functions $f_k(t)$ of the curve satisfy the property $\sum_{k=1}^{n} f_k(t) = 1$ for $t \in [0,1]$.

**Theorem 1:** Let $C$ be a curve defined by $\sum_{k=1}^{n} f_k(t)x_k$ where $t \in [0,1]$. If the basis functions $f_k(t)$ are a partition of unity that is $\sum_{k=1}^{n} f_k(t) = 1$ for $t \in [0,1]$, then $C$ is affine invariant, i.e. for any affine transform $W = TL$ where $L$ is a linear transform and $T$ is a translation by $\mathbf{u}$, we have:

$$W \left( \sum_{k=1}^{n} f_k(t)x_k \right) = \sum_{k=1}^{n} f_k(t)W(x_k)$$

Proof left as homework 🤔
Convex Hull

- Another useful property is that if a curve lies within its convex hull which can be defined as follows:

**Definition 12:** Let \( \{x_1, x_2, \ldots, x_n\} \) be a set of points in the \( d \)-dimensional Euclidean space \( \mathbb{R}^d \) and \( a_1, a_2, \ldots, a_n \) be real numbers, then \( \sum_{i=1}^{n} a_i x_i = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n \) is called a **linear combination** of \( x_1, x_2, \ldots, x_n \). If \( \sum_{i=1}^{n} a_i = 1 \), then this is called an **affine combination** of \( x_1, x_2, \ldots, x_n \), if \( \sum_{i=1}^{n} a_i = 1 \) and \( a_i \geq 0 \), then this is called a **weighted average** of \( x_1, x_2, \ldots, x_n \).

**Definition 13:** Let \( A \) be a set of points in \( \mathbb{R}^d \). The set \( A \) is **convex** if and only if for any two points \( x, y \in A \), the line segment joining \( x \) and \( y \) is entirely in \( A \).
Convex Hull

**Definition 14:** The convex hull of \( A \) is the smallest convex set containing \( A \), hence the convex hull of the set \( A \) is the set of points that are weighted averages of points in that set, thus:

\[
\text{chull}(A) = \left\{ x : x = \sum_{i=1}^{n} a_i x_i, x_i \in A, \sum_{i=1}^{n} a_i = 1, a_i \geq 0 \right\}
\]

Where \( A = \{x_1, x_2, \ldots, x_n\} \).
Interpolation

• In **numerical analysis**, *interpolation* is a method of constructing new points within the range of a discrete set of known points.

• From the **engineering** point of view, we often has a number of points, obtained by sampling or experimentation, and we wish to construct a function which closely fits these points, this is called curve fitting where interpolation is a special case, in which the function must go exactly through the known points.

• Another way to define interpolation is from its **linguistic** meaning, *inter* means between and *pole* mean points or nodes, hence any method of calculating a new point between two or more known points is called interpolation.
n-degree Interpolation

When we n-degree interpolation, we need \( n+1 \) known points, hence we will be given a set of control points, \( x_0, x_1, x_2, \ldots, x_n \) (indexing now starts from 0 rather than 1 to have \( n+1 \) points).

According to Definition 10, a curve can be defined as or generated by \( \sum_{k=1}^{n} f_k(t)x_k \) where \( f_k(t) \) are the basis functions of the curve, hence different basis functions will lead to different curves.

Curve generation can be thought of as an interpolation process where the control points are interpolated to generate points whose locus is the curve.

The simplest form of interpolation is Lagrange interpolation
**Lagrange Interpolation**

**Definition 15:** Let \( x_0, x_1, x_2, \ldots, x_n \) be a set of control points, Lagrange interpolation of these points is given by:

\[
x(t) = \sum_{k=0}^{n} L^n_k(t)x_k
\]

With

\[
L^n_k(t) = \prod_{\substack{j=0 \atop j \neq k}}^{n} \frac{t - t_j}{t_k - t_j}
\]

Where \( t_j \) are the parameter values at which the point \( x_k \) should be interpolated.

**Joseph-Louis Lagrange,** Italian-born mathematician and astronomer, who lived most of his life in Prussia and France, making significant contributions to all fields of analysis, to number theory, and to classical and celestial mechanics.
Example: Uniform Lagrange Interpolation

The coordinates for interpolation are given by the following controlling points, parameter values at which the control points should be interpolated are chosen to be $t_j = j$, this is called uniform Lagrange interpolation:

\[x_0 = (1,1)^T \quad t_0 = 0\]
\[x_1 = (2,3)^T \quad t_1 = 1\]
\[x_2 = (4,-1)^T \quad t_2 = 2\]
\[x_3 = (4.6,1.5)^T \quad t_3 = 3\]
Example: Uniform Lagrange Interpolation

Now, let’s define the Lagrange basis functions with $n = 3$ (degree of interpolation) and $k = 0, 1, 2, 3$

$$L_0^3(t) = \prod_{j=0}^{3} \frac{t - t_j}{t_0 - t_j} = \frac{(t - t_1)(t - t_2)(t - t_3)}{(t_0 - t_1)(t_0 - t_2)(t_0 - t_3)} = \frac{(t - 1)(t - 2)(t - 3)}{(0 - 1)(0 - 2)(0 - 3)}$$

$$= -\frac{1}{6}(t - 1)(t - 2)(t - 3) = -\frac{1}{6}t^3 + t^2 - \frac{11}{6}t + 1$$

$$L_1^3(t) = \prod_{j=0}^{3} \frac{t - t_j}{t_1 - t_j} = \frac{(t - t_0)(t - t_2)(t - t_3)}{(t_1 - t_0)(t_1 - t_2)(t_1 - t_3)} = \frac{(t - 0)(t - 2)(t - 3)}{(1 - 0)(1 - 2)(1 - 3)}$$

$$= \frac{1}{2}t(t - 2)(t - 3) = \frac{1}{2}t^3 - \frac{5}{2}t^2 + 3t$$

$$L_2^3(t) = \prod_{j=0}^{3} \frac{t - t_j}{t_2 - t_j} = \frac{(t - t_0)(t - t_1)(t - t_3)}{(t_2 - t_0)(t_2 - t_1)(t_2 - t_3)} = \frac{(t - 0)(t - 1)(t - 3)}{(2 - 0)(2 - 1)(2 - 3)}$$

$$= -\frac{1}{2}t(t - 1)(t - 3) = -\frac{1}{2}t^3 + 2t^2 - \frac{3}{2}t$$

$$L_3^3(t) = \prod_{j=0}^{3} \frac{t - t_j}{t_3 - t_j} = \frac{(t - t_0)(t - t_1)(t - t_2)}{(t_3 - t_0)(t_3 - t_1)(t_3 - t_2)} = \frac{(t - 0)(t - 1)(t - 2)}{(3 - 0)(3 - 1)(3 - 2)}$$

$$= \frac{1}{6}t(t - 1)(t - 2) = \frac{1}{6}t^3 - \frac{1}{2}t^2 + \frac{1}{3}t$$
Example: Uniform Lagrange Interpolation

Basis functions for uniform Lagrange interpolation of degree 3.
Example: Uniform Lagrange Interpolation

Let's check for the affine invariance, that is if \( \sum_{k=0}^{n} L_k^n(t) = 1 \)

\[
L_0^3(t) + L_1^3(t) + L_2^3(t) + L_3^3(t) \\
= \left( -\frac{1}{6} t^3 + t^2 - \frac{11}{6} t + 1 \right) + \left( \frac{1}{2} t^3 - \frac{5}{2} t^2 + 3t \right) + \left( -\frac{1}{2} t^3 + 2t^2 - \frac{3}{2} t \right) \\
+ \left( \frac{1}{6} t^3 - \frac{1}{2} t^2 + \frac{1}{3} t \right) = 1
\]

Hence the functions \( L_k^n(t) \) are a partition of unity, thus curves produced by Lagrange interpolation are affine invariant.

The curve will be defined by;

\[
C: \quad x(t) = L_0^3(t)x_0 + L_1^3(t)x_1 + L_2^3(t)x_2 + L_3^3(t)x_3 \\
= \left( -\frac{1}{6} t^3 + t^2 - \frac{11}{6} t + 1 \right)x_0 + \left( \frac{1}{2} t^3 - \frac{5}{2} t^2 + 3t \right)x_1 \\
+ \left( -\frac{1}{2} t^3 + 2t^2 - \frac{3}{2} t \right)x_2 + \left( \frac{1}{6} t^3 - \frac{1}{2} t^2 + \frac{1}{3} t \right)x_3
\]
Example: Uniform Lagrange Interpolation
Let’s do it ...

%% our variables
%% parameter
syms t;
%% controlling points (four points)
syms x0 x1 x2 x3;

%% basis functions (cubic)
L0 = expand((-1/6)*(t-1)*(t-2)*(t-3));
L1 = expand((1/2)*(t-0)*(t-2)*(t-3));
L2 = expand((-1/2)*(t-0)*(t-1)*(t-3));
L3 = expand((1/6)*(t-0)*(t-1)*(t-2));
suml = L0 + L1 + L2 + L3;

%% the equation of the curve
x = L0*x0 + L1*x1 + L2*x2 + L3*x3;

%% generating the curve
X = [];
i = 0;
for t = 0 : 0.01 : 3
    % evaluating the current point
    cur_x = eval(x);
    i = i + 1;
    X(:,i) = cur_x;

    % evaluating the individual basis functions
    basisL0(i) = eval(L0);
    basisL1(i) = eval(L1);
    basisL2(i) = eval(L2);
    basisL3(i) = eval(L3);
end

%% the actual values of the controlling points
x0 = [1 1]';
x1 = [2 3]';
x2 = [4 -1]';
x3 = [4.6 1.6]';
control_pts = [x0 x1 x2 x3];
Next ...

- However Lagrange interpolation does not satisfy the convex hull property.

- Typically, we would like the curve to be more confined, i.e. the area of the convex hull of the curve should not be much greater than the area of the convex hull of the control points.

- Next we will discuss some of the more popular curves and how they may be used to interpolate points.
Bézier Curves

Bézier curves are one of the most popular representations for curves.

Pierre Étienne Bézier (September 1, 1910 – November 25, 1999) was a French engineer and patentor (but not the inventor) of the Bézier curves and Bézier surfaces that are now used in most computer-aided design and computer graphics systems.
Bézier Curves

Definition 16: Let \( \mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \) be a set of control points, a Bézier curve of degree \( n \) is given by:

\[
\mathbf{x}(t) = \sum_{k=0}^{n} B_k^n(t) \mathbf{x}_k \quad t \in [0,1]
\]

where the basis functions \( B_k^n(t) \) are the Bernstein polynomials defined by:

\[
B_k^n(t) = \binom{n}{k} t^k (1 - t)^{n-k}
\]

where \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \)

Bézier curves interpolate the end points \( \mathbf{x}_0 \) and \( \mathbf{x}_n \), that is it connects the end points in a fashion directed by in-between control points, which do not lie on the curve, this is called endpoint interpolation property.
Bézier Curves

Definition 17: Given two control points $x_0, x_1$, a Linear Bézier curve is simply a straight line between those two points, the curve is given by:

$$x(t) = (1 - t)x_0 + tx_1, \quad t \in [0,1]$$

The $t$ in the function for a linear Bézier curve can be thought of as describing how far $x(t)$ is from $P_0$ to $P_1$. For example when $t=0.25$, $x(t)$ is one quarter of the way from point $P_0$ to $P_1$. As $t$ varies from 0 to 1, $x(t)$ describes a curved line from $P_0$ to $P_1$. 
**Bézier Curves**

**Definition 18:** A quadratic Bézier curve is the path traces by the function \( x(t) \) given three control points \( x_0, x_1, x_2 \):

\[
x(t) = (1 - t)^2x_0 + 2t(1 - t)x_1 + t^2x_2, \quad t \in [0,1]
\]

A quadratic Bézier curve is also a parabolic segment.

For quadratic Bézier curves one can construct intermediate points \( Q_0 \) and \( Q_1 \) such that as \( t \) varies from 0 to 1: Point \( Q_0 \) varies from \( P_0 \) to \( P_1 \) and describes a linear Bézier curve. Point \( Q_1 \) varies from \( P_1 \) to \( P_2 \) and describes a linear Bézier curve. Point \( x(t) \) varies from \( Q_0 \) to \( Q_1 \) and describes a quadratic Bézier curve.
Bézier Curves

**Definition 19:** Four control points $x_0, x_1, x_2, x_3$ in the plane or in three-dimensional space define a **cubic Bézier curve**. The curve starts at $x_0$ going toward $x_1$ and arrives at $x_3$ coming from the direction of $x_2$, usually it will not pass through $x_1$ or $x_2$, these points are only there to provide directional information, the parametric form of the curve is:

$$x(t) = (1 - 3t + 3t^2 - t^3)x_0 + (3t - 6t^2 + 3t^3)x_1 + (3t^2 - 3t^3)x_2 + t^3x_3, \quad t \in [0,1]$$

For higher-order curves one needs correspondingly more intermediate points. For cubic curves one can construct intermediate points $Q_0, Q_1, Q_2$ that describe linear Bézier curves, and points $R_0, R_1$ that describe quadratic Bézier curves. For fourth-order curves one can construct intermediate points $Q_0, Q_1, Q_2, Q_3$ that describe linear Bézier curves, points $R_0, R_1, R_2$ that describe quadratic Bézier curves, and points $S_0, S_1$ that describe cubic Bézier curves.
**Bézier Curves**

**Definition 20:** Let $x_0, x_1, x_2, \ldots, x_n$ be a set of control points, a Bézier curve of degree $n$ given by:

$$x(t) = \sum_{k=0}^{n} B^n_k(t) x_k \quad t \in [0,1]$$

can be expressed recursively as follows: Let $x_{x_0x_1\ldots x_n}(t)$ denote the Bézier curve denoted by the control points $x_0, x_1, x_2, \ldots, x_n$, then,

$$x(t) = x_{x_0x_1\ldots x_n}(t) = (1-t)x_{x_0x_1\ldots x_{n-1}}(t) + tx_{x_1x_2\ldots x_n}(t)$$

Hence, the Bézier curve of degree $n$ is a linear interpolation between two Bézier curves of degree $n-1$.

**Definition 21:** Let $x_0, x_1, x_2, \ldots, x_n$ be a set of control points of the Bézier curve, the polygon formed by connecting the Bézier points with lines, starting with $x_0$ and finishing with $x_n$, is called the Bézier polygon. The convex hull of the Bézier polygon contains the Bézier curve.
Bernstein Polynomials Properties

**Theorem 2:** The Bernstein polynomials are a partition of unity, i.e. \( \sum_{k=0}^{n} B_k^n(t) = 1 \), hence Bézier curves are affine invariant.

Proof left as homework 😞

**Theorem 3:** The Bernstein polynomials \( B_k^n(t) \) are defined such that \( 0 \leq B_k^n(t) \leq 1 \) for \( t \in [0,1] \). A point of the Bézier curve \( \mathbf{x}(t) = \sum_{k=0}^{n} B_k^n(t) \mathbf{x}_k \) is thus a weighted average of the points \( \mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \). The convex hull of the curve \( \mathbf{x}(t) \) is the set of all weighted averages of \( \mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \). The Bézier curve thus lies in the convex hull of the points \( \mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \), where the convex hull is defined by a polygon created from these points.
Example

• The most popular Bézier curves are Bézier curves of degree 3. Let the control points be given as follows:

\[ x_0 = (1,1)^T \]
\[ x_1 = (2,3)^T \]
\[ x_2 = (4,-1)^T \]
\[ x_3 = (4.6,1.5)^T \]

Now, let's define the Bernstein polynomials, i.e. Bézier basis functions with \( n = 3 \) and \( k = 0,1,2,3 \)

\[
B_0^3(t) = \binom{3}{0} t^0 (1 - t)^{3-0} = \frac{3!}{0! \cdot (3 - 0)!} (1 - t)^3 = (1 - t)^3 = 1 - 3t + 3t^2 - t^3
\]

\[
B_1^3(t) = \binom{3}{1} t^1 (1 - t)^{3-1} = \frac{3!}{1! \cdot (3 - 1)!} t(1 - t)^2 = 3t(1 - t)^2 = 3t - 6t^2 + 3t^3
\]

\[
B_2^3(t) = \binom{3}{2} t^2 (1 - t)^{3-2} = \frac{3!}{2! \cdot (3 - 2)!} t^2(1 - t) = 3t^2(1 - t) = 3t^2 - 3t^3
\]

\[
B_3^3(t) = \binom{3}{3} t^3 (1 - t)^{3-3} = \frac{3!}{3! \cdot (3 - 3)!} t^3 = t^3
\]
Example

Basis functions for Bézier curves of degree 3.
Example

- The curve will be defined by;

\[ c(t) = B_0^3(t)x_0 + B_1^3(t)x_1 + B_2^3(t)x_2 + B_3^3(t)x_3 \]
\[ = (1 - 3t + 3t^2 - t^3)x_0 + (3t - 6t^2 + 3t^3)x_1 + (3t^2 - 3t^3)x_2 + t^3x_3 \]

Convex hull of the curve

Convex hull of the control points

Bézier curve of four points and its convex hull
Let’s do it …

%% our variables
%% parameter
syms t;
%% controlling points (four points)
syms x0 x1 x2 x3;

%% basis functions (cubic)
B30 = expand((1-t)^3);
B31 = expand(3*t*[(1-t)^2]);
B32 = expand(3*[t^2]*[(1-t)];
B33 = t^3;
sumB = B30 + B31 + B32 + B33;

%% the equation of the curve
x = B30*x0 + B31*x1 + B32*x2 + B33*x3;

%% the actual values of the controlling points
x0 = [1 1]';
x1 = [2 3]';
x2 = [4 -1]';
x3 = [4.6 1.5]';
control_pts = [x0 x1 x2 x3];

%% generating the curve
X = [];
i = 0;
for t = 0 : 0.01 : 1
    % evaluating the current point
    cur_x = eval(x);
i = i + 1;
    X(:,i) = cur_x;
end
Derivatives of Bézier Curves

Theorem 4: The derivative of a Bézier curve is also a Bézier curve, furthermore, we have:

\[ x'(0) = n(x_1 - x_0) \text{ and } x'(1) = n(x_n - x_{n-1}) \]

Thus we have immediate form of the derivative at the end points. These are the tangents to the curve at the end points. This will be useful in defining curves that are piecewise continuous.

Homework

Differentiate Bézier curve and proof Theorem 4
Piecewise Continuous Bézier Curves

- We can construct Bézier curves of arbitrary degree, however it becomes more difficult to control the curves since the Bézier curve is only guaranteed to interpolate end points. Instead we can create several Bézier that are piecewise continuous.

Definition 22: Piecewise continuous curves are defined to be continuous curves, where there is also continuity at the points where they join.

- There are different kinds of continuity which can be considered.

Definition 23: Let \( k \geq 0 \), a function \( f \) is \( C^k \) continuous if \( f \) has the \( k^{\text{th}} \) derivative defined and continuous everywhere in the domain of \( f \). \( C^0 \) continuity is simply the usual definition of continuity. \( f \) is \( C^\infty \) continuous if \( f \) is \( C^k \) continuous for all \( k \geq 0 \).

Definition 24: Let \( f \) be a continuous function, let \( t = t(u) \) be a continuous and strictly increasing function, let \( g(u) = f(t(u)) \), if \( g(u) \) has a continuous, non-zero first derivative everywhere in its domain, then \( f \) is said to be \( G^1 \) continuous.
Piecewise Continuous Bézier Curves

We can obtain $C^1$ continuity between two Bézier curves $q(t)$ and $r(t)$ defined as:

$$q(t) = \sum_{k=0}^{n} B^n_k(t)q_k$$
and
$$r(t) = \sum_{k=0}^{n} B^n_k(t)r_k$$

Assuming that $q(t)$ will occur before $r(t)$, by requiring $q'(1) = r'(0)$, using Theorem 4, we will have:

$$q'(1) = n(q_n - q_{n-1})$$
and
$$r'(0) = n(r_1 - r_0)$$

thus we have

$$q_n - q_{n-1} = r_1 - r_0$$

For $G^1$ continuity, we require that:

$$q_n - q_{n-1} = s(r_1 - r_0), s > 0$$

This can be adapted to Bézier curves of different degrees.
Piecewise Continuous Bézier Curves

**Theorem 5:** A piecewise smooth curve $C^1$ from $m$ Bézier curves, $q^i(t), i = 1, 2, \ldots, m$ of degree $n$ by requiring that:

$$q^i_n - q^i_{n-1} = q^{i+1}_1 - q^{i+1}_0 ; i = 1, 2, \ldots, m$$

The continuous curve can then be defined by:

$$Q(t) = \begin{cases} 
q^1(t) & t \in [0, \frac{1}{m}) \\
q^2(t) & t \in \left[\frac{1}{m}, \frac{2}{m}\right) \\
& \vdots \\
q^m(t) & t \in \left[\frac{m-1}{m}, 1\right]
\end{cases}$$

where the control points of $Q(t)$ are simply the control points of $q^i(t), i = 1, 2, \ldots, m$ with the restriction mentioned above.
Piecewise Continuous Bézier Curves

Definition 16 gives the equation of a Bézier curve which starts at \( t = 0 \) and ends at \( t = 1 \). It is useful, especially when fitting together a string of Bézier curves, to allows an arbitrary parameter interval.

**Definition 24:** Let \( x_0, x_1, x_2, \ldots, x_n \) be a set of control points, a Bézier curve of degree \( n \) defined over an interval \( t \in [t_0, t_1] \) is given by:

\[
x(t) = \sum_{k=0}^{n} B_k^n(t)x_k, \quad t \in [t_0, t_1]
\]

where the basis functions \( B_k^n(t) \) are the Bernstein polynomials defined over \( t \in [t_0, t_1] \) given by:

\[
B_k^n(t) = \binom{n}{k} \left( \frac{t - t_0}{t_1 - t_0} \right)^k \left( \frac{t_1 - t}{t_1 - t_0} \right)^{n-k}
\]

where \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \).
Example

Let the control points of $q^1$ be given as follows

\[ q_0^1 = (0,1)^T \]
\[ q_1^1 = (0.5,1)^T \]
\[ q_2^1 = (1,0.5)^T \]
\[ q_3^1 = (1,0)^T \]

Let the control points of $q^2$ be given as follows

\[ q_0^2 = (1,0)^T \]
\[ q_1^2 = (1,-0.5)^T \]
\[ q_2^2 = (2,0)^T \]
\[ q_3^2 = (2,0.5)^T \]

Note that $q_3^1 - q_2^1 = q_1^2 - q_0^2$
Example

The curve will be defined as follows:

\[ Q(t) = \begin{cases} 
q^1(t) & t \in [0, \frac{1}{2}) \\
q^2(t) & t \in \left[ \frac{1}{2}, 1 \right] 
\end{cases} \]

where:

\[ q^i(t) = \sum_{k=0}^{3} B_{k,i}^3(t) q^i_k = B_{0,i}^3(t) q^i_0 + B_{1,i}^3(t) q^i_1 + B_{2,i}^3(t) q^i_2 + B_{3,i}^3(t) q^i_3 \]

With:

\[ B_{k,i}^n(t) = \binom{n}{k} \left( \frac{t - t_{i-1}}{t_i - t_{i-1}} \right)^k \left( \frac{t_i - t}{t_i - t_{i-1}} \right)^{n-k} \]
Example

Let’s find the basis functions for the first Bézier curve, where \( i = 1, n = 3 \) and \( t \in [0, \frac{1}{2}] \), hence \( t_{i-1} = t_0 = 0 \) and \( t_i = t_1 = \frac{1}{2} \).

\[
B_{0,1}^3(t) = \binom{3}{0} \left( \frac{t - 0}{\frac{1}{2} - 0} \right)^0 \left( \frac{1}{2} - t \right)^3 = 0!
\]

\[
= (1 - 6t + 12t^2 - 8t^3)
\]

\[
B_{1,1}^3(t) = \binom{3}{1} \left( \frac{t - 0}{\frac{1}{2} - 0} \right)^1 \left( \frac{1}{2} - t \right)^3 = \frac{3!}{1! (3 - 1)!}
\]

\[
= 2t(1 - 2t)^2 = 3(2t)(1 - 2t)^2
\]

\[
= 6t - 24t^2 + 24t^3
\]

\[
B_{2,1}^3(t) = \binom{3}{2} \left( \frac{t - 0}{\frac{1}{2} - 0} \right)^2 \left( \frac{1}{2} - t \right)^3 = \frac{3!}{2! (3 - 2)!}
\]

\[
= \frac{3}{2}(2t)(1 - 2t) = 3(2t)^2 (1 - 2t)
\]

\[
= 12t^2 - 24t^3
\]

\[
B_{3,1}^3(t) = \binom{3}{3} \left( \frac{t - 0}{\frac{1}{2} - 0} \right)^3 \left( \frac{1}{2} - t \right)^3 = \frac{3!}{3! (3 - 3)!}
\]

\[
= (2t)^3 = (2t)^3 = 8t^3
\]
Example

Thus,

\[ q^1(t) = B_{0,1}^3(t)q_0^1 + B_{1,1}^3(t)q_1^1 + B_{2,1}^3(t)q_2^1 + B_{3,1}^3(t)q_3^1 \]

\[ = (1 - 6t + 12t^2 - 8t^3)q_0^1 + (6t - 24t^2 + 24t^3)q_1^1 + (12t^2 - 24t^3)q_2^1 + (8t^3)q_3^1 \]
Example

Now, let’s find the basis functions for the second Bézier curve, where \( i = 2, n = 3 \) and \( t \in \left[ \frac{1}{2}, 1 \right] \), hence \( t_{i-1} = t_1 = \frac{1}{2} \) and \( t_i = t_2 = 1 \),

\[
B_{0,2}^3(t) = \binom{3}{0} \left( \frac{t - \frac{1}{2}}{1 - \frac{1}{2}} \right)^0 \left( \frac{1 - t}{1 - \frac{1}{2}} \right)^{3-0} = \frac{3!}{0!(3-0)!} (2 - 2t)^3 = (2 - 2t)^3 \\
= 8 - 24t + 24t^2 - 8t^3
\]

\[
B_{1,2}^3(t) = \binom{3}{1} \left( \frac{t - \frac{1}{2}}{1 - \frac{1}{2}} \right)^1 \left( \frac{1 - t}{1 - \frac{1}{2}} \right)^{3-1} = \frac{3!}{1!(3-1)!} (2t - 1)(2 - 2t)^2 = 3(2t - 1)(2 - 2t)^2 \\
= -12 + 48t - 60t^2 + 24t^3
\]

\[
B_{2,2}^3(t) = \binom{3}{2} \left( \frac{t - \frac{1}{2}}{1 - \frac{1}{2}} \right)^2 \left( \frac{1 - t}{1 - \frac{1}{2}} \right)^{3-2} = \frac{3!}{2!(3-2)!} (2t - 1)^2(2 - 2t) = 3(2t - 1)^2(2 - 2t) \\
= 6 - 30t + 48t^2 - 24t^3
\]

\[
B_{3,2}^3(t) = \binom{3}{3} \left( \frac{t - \frac{1}{2}}{1 - \frac{1}{2}} \right)^3 \left( \frac{1 - t}{1 - \frac{1}{2}} \right)^{3-3} = \frac{3!}{3!(3-3)!} (2t - 1)^3 = (2t - 1)^3 \\
= -1 + 6t - 12t^2 + 8t^3
\]
Example

Thus,

\[ q^2(t) = B_{0,2}^3(t)q_0^2 + B_{1,2}^3(t)q_1^2 + B_{2,2}^3(t)q_2^2 + B_{3,2}^3(t)q_3^2 \]
Example

\[ Q(t) = \begin{cases} 
  q^1(t) & t \in [0, \frac{1}{2}) \\
  q^2(t) & t \in [\frac{1}{2}, 1] 
\end{cases} \]
%% our variables
% parameter
syms t;
% controlling points (four points)
% first piece
syms q10 q11 q12 q13;
% second piece
syms q20 q21 q22 q23;

%% basis functions (cubic)
% for the first piece
B30_1 = expand((1-2*t)^3);
B31_1 = expand(3*(2*t)*(1-2*t)^2);
B32_1 = expand(3*((2*t)^2)*(1-2*t));
B33_1 = expand((2*t)^3);

% for the second piece
B30_2 = expand((2-2*t)^3);
B31_2 = expand(3*(2*t-1)*((2-2*t)^2));
B32_2 = expand(3*((2*t-1)^2)*(2-2*t));
B33_2 = expand((2*t-1)^3);

%% the equation of the curve
% the first piece
q1 = B30_1*q10 + B31_1*q11 + B32_1*q12 + B33_1*q13;
% the second piece
q2 = B30_2*q20 + B31_2*q21 + B32_2*q22 + B33_2*q23;
%% the actual values of the controlling points
%% the first piece
q10 = [0  1]';
q11 = [0.5  1]';
q12 = [1  0.5]';
q13 = [1  0]';

% the second piece
q20 = [1  0]';
q21 = [1  -0.5]';
q22 = [2  0]';
q23 = [2  0.5]';

control_pts = [q10 q11 q12 q13 q20 q21 q22 q23];

%% generating the curve
Q = [];
i = 0;
for t = U : U.01 : 1
  % evaluating the current point
  if t < 0.5
    cur_q = eval(q1);
  else
    cur_q = eval(q2);
  end
  i = i + 1;
  Q(:,i) = cur_q;

  % evaluating the individual basis functions
  basisR3N_1(i) = eval(R3N_1);
  basisB31_1(i) = eval(B31_1);
  basisB32_1(i) = eval(B32_1);
  basisB33_1(i) = eval(B33_1);

  % evaluating the individual basis functions
  basisD3O_2(i) = eval(D3O_2);
  basisB31_2(i) = eval(B31_2);
  basisB32_2(i) = eval(B32_2);
  basisB33_2(i) = eval(B33_2);
end
Next ...

- Is this the optimal way to render a Bezier curve, I don’t think so ...

- We will investigate methods commonly used to efficiently compute curve’s points.

- In addition, we will visit B-splines and generalize this to surfaces ....
Thank You