A Tutorial on Registration

Mutual Information Based Registration

Probability Review

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Experiment

• **Experiment** – in probability theory refers to a process whose outcome is not known in advance with certainty.

  – E.g., Suppose a coin is tossed 10 times. How many times will we get “heads”?

  – Experience tells us that, if the coin is fair, we will see on average 5 “heads”; we can arrive at this result by performing the experiment many times and noting down the observations.

  – Instead of performing the experiments, we can use probability theory to develop a model of any system that yields uncertain measurements.
Sample Space, Outcome, Event !!!

• **Sample space** ($\Omega$) – is the collection of all possible outcomes of a system or an experiment.
  
  – E.g., Rolling a six-sided dice can be modeled with the sample space $\Omega = \{1, 2, 3, 4, 5, 6\}$
  
  – E.g., Model of the number of packets queued at a router that can buffer up to 50 packets would use the sample space $\Omega = \{0, 1, 2, ..., 50\}$

• **Outcome** – is element in the sample space $\Omega$.
  
  – E.g., 2, 4, and 5 are outcomes in the rolling dice example.

• **Event** – is a subset of the sample space
  
  – E.g., $A$ is an event that an even number is rolled: $A = \{2, 4, 6\}$
What is Probability?!!!

• Probability is a measure of how likely it is for an event $A$ to happen.

• We name a probability with a number from 0 to 1.

• If an event is certain to happen, then the probability of the event is 1.

• If an event is certain not to happen, then the probability of the event is 0.
What is Probability?!!!

- If it is **uncertain** whether or not an event will happen, then its probability is some fraction between 0 and 1 (or a fraction converted to a decimal number).

1. What is the probability that the spinner will stop on part A?

2. What is the probability that the spinner will stop on
   (a) An even number?
   (b) An odd number?

3. What fraction names the probability that the spinner will stop in the area marked A?
Prior Probability

• *Prior probability*: the probability before we consider any additional knowledge.

\[ P(A) \]
Conditional Probability

• Sometimes we have partial knowledge about the outcome of an experiment

• Conditional (or Posterior) Probability

• Suppose we know that event B is true

• The probability that A is true given the knowledge about B is expressed by

\[ P(A | B) \]
Conditional Probability

- This rule comes from the fact that for A and B to be true we need B to be true and A to be true given B.
- Joint probability of A and B - \( P(A,B) \) - is a 2-dimensional table with a value in every cell giving the probability of that specific state occurring.
Independence

• Two events $A$ and $B$ are independent of each other if

$$P(A) = P(A | B)$$

i.e. knowing $B$ does not add any information when measuring the probability of $A$, hence

$$P(A,B) = P(A)P(B)$$

• Two events $A$ and $B$ are conditionally independent of each other given an event $C$ if

$$P(A | C) = P(A | B,C)$$
Bayes’ Theorem

• Bayes’ Theorem lets us swap the order of dependence between events.

• We saw that

\[ P(A, B) = P(A | B)P(B) \Rightarrow P(A | B) = \frac{P(A, B)}{P(B)} \]

• Bayes’ Theorem:

\[ P(A | B) = \frac{P(B | A)P(A)}{P(B)} \]

\[ P(A, B) = P(B | A)P(A) \]
Random Variables

• So far, sample space $\Omega$ differs with every problem/experiment we look at.

• Random variables (RV) $X$ allow us to talk about the probabilities of numerical values that are related to the event space.

• Random Variable is a real-valued function which maps the sample space $\Omega$ to the real number for continuous RV, or to the integer numbers for discrete RV.

$$X : \Omega \rightarrow \mathbb{R}$$

$$X : \Omega \rightarrow \mathbb{Z}$$

• Because a RV has a numerical range, we can often do math more easily by working with the values of a RV rather than directly with events and we can define a probability for a random variable as follows:

$$p(x) = p(X = x) = p(A_x) \quad \text{where} \quad A_x = \{\omega \in \Omega : X(\omega) = x\}$$

• The probability of a random variable should satisfy the following:

$$\sum_x p(x) = 1 \quad \text{and} \quad 0 \leq p(x) \leq 1$$
Statistics of Random Variable

- The *Expectation* is the *mean* or *average* of a RV

\[
E(X) = \sum_x x p(x) = \mu
\]

- The *variance* of a RV is a measure of whether the values of the RV tend to be consistent over trials or to vary a lot.

\[
Var(X) = E((X - E(X))^2)
\]

\[
= E(X^2) - E^2(X) = \sigma^2
\]

- \( \sigma \) is the *standard deviation*
Joint Probability

- Joint probability of two random variables $X$ and $Y$, $p(X,Y)$, is probability that $X$ and $Y$ simultaneously assume particular values.

  - If $X$, $Y$ independent, $p(X, Y) = p(X)p(Y)$

- Roll die, toss coin

  - $p(X = 3, Y = \text{heads}) = p(X = 3)p(Y = \text{heads}) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$
References

• http://www-nlp.stanford.edu/fsnlp/mathfound/rosario-math-foundations.ppt
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