Chapter 1

An Overview of Analog Signals and Systems

1.1 Introduction

A number of important signal processing tasks can be modeled as an input-output system; that is, the input signal is mapped into a desired output signal by a certain mapping function or operator. This is illustrated in Figure 1.1. The system function $h(\cdot)$ is the mapping from the input $x(\cdot)$ into the output $y(\cdot)$ and it can take a variety of forms.

Example 1:
Suppose the output $y(\cdot)$ is the derivative of the input $x(\cdot)$; i.e., $y(t) = \frac{d}{dt} x(t)$, $t$ is a variable (e.g., time). This relationship can be modeled as an input-output system as shown in Figure 1.2.

Figure 1.2: Modeling $y(t) = \frac{d}{dt} x(t)$ as an input-output system.

Figure 1.3 shows two input signals $x(t)$ and the corresponding output signals $y(t)$ for the system in Figure 1.2.

A system can be connected to another system in various forms. For example, two systems connected in cascade (series) represent two successive operations. In block diagram, this is
shown in Figure 1.4. The cascade connection in Figure 1.4(a) can be expressed, equivalently, as one system as in Figure 1.4(b). We will study specific examples in which the system function $h(\cdot)$ is easily obtained from $h_1(\cdot)$ and $h_2(\cdot)$. Another system interconnection is the parallel interconnection, illustrated in Figure 1.5. The output $y(\cdot)$ is obtained from combining the outputs $x_1(\cdot)$ and $x_2(\cdot)$ of systems 1 and 2 by some arithmetic or algebraic operation $O$. For example, $y(t)$ equals $x_1(t) - x_2(t)$ when $O$ represents subtraction.

A third system interconnection combines the cascade and parallel interconnection. This
is illustrated in Figure 1.6. Another example, Figure 1.7, shows a system realization for

\[ y(t) = \sqrt{2x(t)} - x^2(t). \]

Figure 1.6: Combination of cascade and parallel interconnections

Example 2:

\[ y(t) = \sqrt{2x(t)} - x^2(t) \]

can be modeled as follows:

Figure 1.7: Modeling of as an interconnection of systems.

Finally, a system can have feedback interconnections as shown in Figure 1.8.
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Figure 1.8: A feedback interconnection.

**Example 3**: Filtering

![Filter Diagram](image)

A filter is an example of system operators that alters the frequency spectrum (to be studied later) of an input signal to achieve a desired output.

As an example of a system, a filter can be used and the output signal \( y(\cdot) \) is obtained by altering the spatial or frequency characteristics of the input signal \( x(\cdot) \). The filter \( h(\cdot) \) can take various forms, and can be expressed in terms of interconnections of other filters (systems) as in Figures 1.6, 1.8.

An example of a filter, Figure 1.9 shows an ideal lowpass filter that retains a specific low-frequency range in the input \( x(\cdot) \). In Figure 1.9, the quantities \( X, H, \) and \( Y \), are the Fourier transforms of the quantities \( x(t), h(t), \) and \( y(t) \), respectively. We will study the Fourier transform shortly.

Other forms of filters include the high-pass, bandpass, and bandstop filters. Their Fourier transforms are shown below in Figure 1.10.

In conclusion, filtering is a very important tool in signal processing. A filter is a system operator that maps the input to the output to achieve a desired result. In order to understand the issue of filtering, we examine some concepts in systems theory.
1.2 One-dimensional continuous-time Systems

1.2.a Definitions

A system is an operator or a mapping, $L(\cdot)$ from an input (or a set of inputs) to an output (or a set of outputs). Figure 1.2a is a block diagram of a multi-input multi-output system.
The number of inputs $n$ need not to be equivalent to the number of outputs $m$.

![Block diagram of a multi-input multi-output system](image)

Figure 1.11: A block diagram of a multi-input multi-output system.

Systems are usually classified into continuous-time or discrete-time systems, depending on whether the inputs are continuous-time or discrete time. Therefore, we say a system is continuous if its inputs are continuous-time signals. Similarly, a system is discrete if its inputs are discrete-time signals (sequences).

The theory of discrete-time systems can be studied independent of the theory of continuous-time systems. However, some insight can be obtained in the treatment of discrete-time systems, if we review a few elements of continuous-time systems theory.

Systems (in general) can be classified in terms of other properties, such as: linearity, causality, time-invariance, stability. We will define these properties and discuss their significance for both continuous-time and discrete-time systems.

**Definition:** A system is called a *memoryless* system if its output for each value of time depends on the input at that same time.

**Example 4:**
(a) \( y(t) = x(t) \); memoryless system
(b) \( y(t) = 9x(t) \); memoryless system
(c) \( y(t) = x(t - 1) \); a system with memory
(d) \( y(t) = K \int_{-\infty}^{t} x(\tau)d\tau, \ K \) is a const.; a system with memory

Definition: A system is causal (non-anticipative) if the output at any time depends only on values of the input at the present time and in the past. In other words, the present output does not depend on future inputs.

Example 5:

```
\[ y(t) = 6x(t) + 7x(t - 9); \] causal system
\[ y(t) = 25x^2(t) + \cos(t - 1); \] causal system
\[ y(t) = 7x(t); \] causal system
\[ y(t) = 7; \] noncausal system
\[ y(t) = \sin(t + 4); \] noncausal system
\[ y(t) = x(t) + x(t - 5) + 1; \] causal system
\[ y(t) = \int_{-\infty}^{t} x(\tau)d\tau; \] causal system.
```

Fact:

All memoryless systems are causal. The converse is not necessarily true.

Proof:

By definition, in memoryless systems, the output at time \( t \) depends only on the input at the same \( t \) which satisfies causality requirements. However, a causal system can have its output depending on past inputs (by definition of causality) which will make such a system non memoryless (or with memory).
1.2. ONE-DIMENSIONAL CONTINUOUS-TIME SYSTEMS

Q.E.D

Example 6:

\[ y(t) = K \int_{-\infty}^{t} x(\tau) d\tau \]  \hspace{1cm} (1.1)

\( y(t) \) describes a causal system since \( y(t) \) depends on \( x(\tau) \), \( \tau \in [-\infty, t] \); but such a system is with memory.

Example 7:

\[ y(t) = 5x^2(t) + 7x(t) \]  \hspace{1cm} (1.2)

describes both causal and memoryless system.

Definition: A system is stable if, for every bounded input, the output is bounded (i.e., finite).

Example 8:

\[ y(t) = 10 \sin(t); \] represents a stable system
\[ y(t) = \int_{-\infty}^{5} x(t) dt; \] might not be bounded for some \( x(t) \).
e.g., if \( x(t) = \text{constant} \), \( y(t) = \infty! \)

Definition: A system is time-invariant if a time shift in the input signal causes the same time shift in the output signal.

Example 9:

(a) \( y(t) = 7x(t); \) is time-invariant
\( \text{Since } y_1(t) = 7x(t - t_o) \) for all \( t_o. \)
\( \equiv y(t - t_o). \)

(b) \( y(t) = 5 + 11x^2(t); \) is time-invariant.
\( \text{Since } y_1(t) = 5 + 11x^2(t - t_o) \)
\( \equiv y(t - t_o) \)
(c) \( y(t) = tx(t); \) is not time-invariant (i.e., time variant).
Since, if we shift \( x(t) \) by a time \( t_0 \)
then \( y_1(t) = tx(t - t_0) \)
\( \neq y(t - t_0) = (t - t_0)x(t - t_0). \)

(d) \( y(t) = t^2 \sin(x(t)); \) is not time-invariant.
If we shift \( x(t) \) by time \( t_0, \)
then \( y_1(t) = t^2 \sin[x(t - t_0)] \)
\( \neq y(t - t_0) = (t - t_0)^2 \sin(x(t - t_0)). \)

(e) \( y(t) = \sin(x(t)); \) is time-invariant.
If we shift \( x(t) \) by time \( t_0, \)
then \( y_1 = \sin(x(t - t_0)) \)
\( = y(t - t_0) \)

\( \diamond \diamond \diamond \)

Definition: A linear system is one that possesses the superposition property. Mathematically, let \( y_1(t) \) be the response of a continuous-time system to \( x_1(t) \)
and let \( y_2(t) \) be the output corresponding to the input \( x_2(t). \) Then the system is linear if:

1. The response (output) to \( x_1(t) + x_2(t) \) is \( y_1(t) + y_2(t); \) additivity property.
2. The response to \( ax_1(t) \) is \( ay_1(t), \) where \( a \) is a constant; scaling or homogeneity property.\( \diamond \)

Note:

1. The above can be illustrated in Figure 1.12.

2. A system can be homogeneous but not additive, and vice versa. In either case
the system will not be linear.
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\[ x_1(t) \xrightarrow{L(\cdot)} y_1(t) \quad x_2(t) \xrightarrow{L(\cdot)} y_2(t) \]

\[ ax_1(t) + bx_2(t) \quad \xrightarrow{L(\cdot)} ay_1(t) + by_2(t) \]

Figure 1.12: Properties of a linear system when \( a \) and \( b \) are constants

**Example 10:**

\[ y(t) = \frac{(x'(t))^2}{x(t)} \]

If \( x(t) = x_1(t) \),
\[ y_1 = \frac{(x'_1(t))^2}{x_1(t)} \]

If \( x(t) = x_2(t) \),
\[ y_2 = \frac{(x'_2(t))^2}{x_2(t)} \]

If \( x(t) = x_1(t) + x_2(t) \),
\[ y(t) = \frac{(x'_1(t) + x'_2(t))^2}{(x_1(t)x_2(t))} \neq y_1(t) + y_2(t) \]

Hence, the system is **nonadditive** and **nonlinear**.

\[ \odot \odot \odot \]

**Note:** In the above system,

If \( x(t) = 5x_1(t) \), then
\[ y_1(t) = \frac{(5x'_1(t))^2}{5x_1(t)} \]
\[ = 5\frac{(x'_1(t))^2}{x_1(t)} \]
\[ = 5y(t), \]

(1.3)
i.e., the system is \textit{homogeneous}.

\textbf{Example 11:}

\[ y(t) = 3x(t) + 1 \]

Homogeneity:

\[
\begin{align*}
\text{If} & \quad x_1(t) = 5x(t), \quad \text{then} \\
y_1(t) & = 15x(t) + 1 \\
& \neq 5y(t)
\end{align*}
\]

Hence, the system is \textit{not homogeneous}. Therefore, the system is \textit{nonlinear}.

\textbf{Additivity:}

\[
\begin{align*}
y_1(t) & = 3x_1(t) + 1 \\
y_2(t) & = 3x_2(t) + 1 \\
y_3(t) & = 3x_3(t) + 1 \\
x_3(t) & = x_1(t) + x_2(t)
\end{align*}
\]

\[
y_3(t) = 3(x_1(t) + x_2(t)) + 1 \\
& \neq y_1(t) + y_2(t)
\]

Hence, the system is \textit{not additive}.

\textbf{Example 12:}

\[ y(t) = 5x(t) \]

Homogeneity:

\[ 5ax(t) = ay(t) \]
Hence, the system is homogeneous.
Additivity:

\[ 5(x_1(t) + x_2(t)) = 5x_1(t) + 5x_2(t) = y_1(t) + y_2(t) \]

Hence, the system is additive as well as homogeneous. Therefore, the system is linear.

Exercise:

Test which of the following systems satisfies some, all, or none of the memoryless, time invariance, causality, stability, and linearity properties.

(a) \( y(t) = e^{x(t)} \)
(b) \( y(t) = \frac{d}{dt}x(t) \)
(c) \( y(t) = x(t - 1) - x(1 - t) \)
(d) \( y(t) = x(t) \sin 6t \)
(e) \( y(t) = \int_{-\infty}^{3t} x(\tau)d\tau \)
(f) \( y(t) = \begin{cases} 0 & \text{if } t < 0 \\ x(t) + x(t - 100) & \text{if } t \geq 0 \end{cases} \)
(g) \( y(t) = x(t/2) \)

Tabulate your results in a table as below.

<table>
<thead>
<tr>
<th>System</th>
<th>Memoryless</th>
<th>Time-invariant</th>
<th>Causal</th>
<th>Stable</th>
<th>Linear</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td></td>
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<tr>
<td>(b)</td>
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<td>(g)</td>
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</tr>
</tbody>
</table>

Special Function Definitions

Definition: The impulse (Dirac-delta) function \( \delta(t) \) is defined in terms of the following properties:
i) \[ \delta(t) = \begin{cases} \text{undefined} & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases} \]

ii) \[ \int_{-\infty}^{\infty} \delta(t) \, dt = 1 \]

iii) \[ \int_{-\infty}^{\infty} \delta(t - t_0) f(t) \, dt = f(t_0) \]
\[ \equiv \int_{-\infty}^{\infty} \delta(t) f(t - t_0) \, dt \]

Example 13:

\[ x(t) = 7t^3 \]

\[ \int_{-\infty}^{\infty} x(t) \delta(t - 3) \, dt = x(3) = 7 \times 27 = 189 \]

Example 14:

\[ x(t) = 18t + e^{-t} + \cos(t) \]

(a) \[ \int_{-\infty}^{\infty} x(t) \delta(t) \, dt = x(0) = 2 \]

(b) \[ \int_{-\infty}^{\infty} \delta(t - 1) x(t) \, dt = 18 + e^{-1} + \cos(1) \]

**Definition:** The unit-step function \( u(t) \) is defined in terms of the following properties.
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\[ u(t) = \begin{cases} 
  1, & t \geq 0 \\
  0, & t < 0 
\end{cases} \]

**Note:**

From distribution theory, we see that \( \frac{d}{dt} u(t) = \delta(t) \).

**Definition:** Consider a system with input \( x(t) \) and output (response) \( y(t) \). The **impulse response** of the system is the output \( y(t) \) when \( x(t) = \delta(t) \). The **step response** of the system is the output \( y(t) \) when \( x(t) = u(t) \). The **sinusoidal response** of the system is the output \( y(t) \) when \( x(t) = \text{sinusoidal function} \) (e.g., \( x(t) = \beta \cos(\alpha t + \theta) \)).

1.2.b Analysis of Linear Systems

**General Concepts**

Recall that we defined linearity in terms of the superposition property; that is, a system is said to be linear if

\[ L(a_1 x_1(t) + a_2 x_2(t)) = a_1 y_1(t) + a_2 y_2(t) = a_1 L(x_1(t)) + a_2 L(x_2(t)) \]

Where \( L(\cdot) \) is the system’s function, \( x(\cdot) \) is the system’s input, \( y(\cdot) \) is the system’s output, and \( a_1 \) and \( a_2 \) are constants. Also stated, the superposition property involves two subproperties: **homogeneity** and **additivity**.

Linear system theory has very rich mathematical foundations. A number of practical problems can be analyzed using this theory. Also, some nonlinear systems can be satisfactorily approximated by linear systems techniques. Most of the issues in image enhancement...
and image restoration can be modeled as linear systems. We will study a number of properties of linear systems in the following pages. First, we will provide a few more examples to further illustrate the concept of linearity.

Example 15:

Consider the system described by the following differential equation.

$$y(t) = \frac{dx(t)}{dt} + \alpha x(t) \quad (1.4)$$

where $\alpha$ is a constant.

$$y_1(t) = \frac{dx_1(t)}{dt} + \alpha x_1(t) \quad (1.5)$$

$$y_2(t) = \frac{dx_2(t)}{dt} + \alpha x_2(t) \quad (1.6)$$

Now let

$$y_3(t) \triangleq a_1 y_1(t) + a_2 y_2(t) \quad (1.7)$$

$$y_3(t) = \frac{dx_3(t)}{dt} + \alpha x_3(t) \quad (1.8)$$

Multiply 1.5 by $a_1$ and 1.6 by $a_2$ and add,

$$y_3(t) = a_1 y_1(t) + a_2 y_2(t) = \left( a_1 \frac{dx_1(t)}{dt} + a_1 \alpha x_1(t) \right) + \left( a_2 \frac{dx_2(t)}{dt} + a_2 \alpha x_2(t) \right)$$

$$\equiv \frac{d}{dt} (a_1 x_1(t) + a_2 x_2(t)) + \alpha (a_1 x_1(t) + a_2 x_2(t))$$

$$\equiv \frac{dx_3(t)}{dt} + \alpha x_3(t) \quad (1.9)$$

Comparing 1.5, 1.6, and 1.9 we see that the definition of linearity is satisfied.

Note to make sure that the additivity property is satisfied, it is assumed that $x(0) = 0$. 

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Exercise:

Show that the system described by

\[ \frac{dx(t)}{dt} + 10x(t) + 5 = y(t) \]

is not linear.

Example 16:

Consider the following system in Figure 1.13 is it linear system?.

We see that:

\[ y_1(t) = x_1(t)\cos 100\pi t \quad (1.10) \]
\[ y_2(t) = x_2(t)\cos 100\pi t \quad (1.11) \]
\[ y_3(t) = x_3(t)\cos 100\pi t \quad (1.12) \]
\[ x_3(t) = a_1 x_1(t) + a_2 x_2(t) \quad (1.13) \]
\[ y_3(t) = (a_1 x_1(t) + a_2 x_2(t))\cos 100\pi t \]
\[ = a_1 x_1(t)\cos 100\pi t + a_2 x_2(t)\cos 100\pi t \]
\[ \equiv a_1 y_1(t) + a_2 y_2(t) \quad (1.14) \]
Hence, from 1.10, 1.11, and 1.14, the system is linear.

Exercise

Show that the system described by

$$y(t) = 10x^2(t) + x(t) \quad (1.15)$$

is not linear.

Hint: Let \( x_1(t) = \cos 100\pi t \) and \( x_2(t) = u(t) \), show that \( x_1(t) + x_2(t) \neq y_1(t) + y_2(t) \).

The Impulse Response of Linear Systems

Consider the linear system where the output is expressed as

$$y(t) = L(x(t)), \quad (1.16)$$

where \( L(\cdot) \) is the system function. Let

$$x(t) = \delta(t - t_1), \quad (1.17)$$

where \( \delta(\cdot) \) is the Dirac-delta function. From the properties of \( \delta(\cdot) \) we know that

$$x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau \quad (1.18)$$

Now, we want to find \( y(t) \) when \( x(t) = \delta(t) \). From 1.16 and 1.18, we have

$$y(t) = L \left( \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau \right).$$

Since \( \delta(t - \tau) \) is the only function of time,

$$y(t) = \int_{-\infty}^{\infty} x(\tau) L(\delta(t - \tau)) d\tau \overset{\Delta}{=} \int_{-\infty}^{\infty} x(\tau) h(t, \tau) d\tau, \quad (1.19)$$

where \( h(t, \tau) = L(\delta(t - \tau)) \) is called the unit impulse-response; which is the output when the input is a unit impulse applied at \( t = \tau \). Figure 1.14 describes the unit impulse-response.
The importance of the impulse response in linear systems will become apparent shortly.

Now, for a linear system to be time-invariant, we must have

\[ h(t, \tau) = h(t - \tau); \quad (1.20) \]

that is, the impulse response is not a function of the time at which the impulse was applied. From 1.19 and 1.20, we see that for a LTI System,

\[ y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau \quad (1.21) \]

which is in the form of the all important convolution integral. The form in 1.21 will also have a lot of consequences which will be examined after we gain experience with evaluating such an integral in the time domain.

**Evaluation of the Convolution Integral**
Consider the integral in 1.21.

\[ y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau \]

Let

\[ t - \tau = \alpha \]
\[ d\tau = -d\alpha, \]
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where \( \tau \) varies from \(-\infty\) to \(\infty\), and \( \alpha \) varies from \(\infty\) to \(-\infty\), therefore,

\[
y(t) = -\int_{-\infty}^{\infty} x(t-\alpha)h(\alpha)d\alpha \equiv \int_{-\infty}^{\infty} x(t-\alpha)h(\alpha)d\alpha
\]

Since \( \tau \) and \( \alpha \) are dummy variables, it is evident that

\[
\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau,
\]

This is written symbolically as

\[
y(t) = x(t) * h(t) \equiv h(t) * x(t);
\]

that is, the convolution is \textit{commutative}.

The expression in 1.22 determines the steps in evaluating the convolution integral. They are as follows:

1. Usually, we have \( t \) as the desired independent variable; i.e., we have \( x(t) \) and \( h(t) \). We want \( y(t) = x(t) * h(t) \). So sketch \( x(\tau) \) and \( h(\tau) \).
2. Flip \( h(\tau) \) (or \( x(\tau) \)).
3. Shift \( h(-\tau) \) (or \( x(-\tau) \)) by \( t \), multiply by \( x(\tau) \) (or \( h(\tau) \)), and integrate.

This is illustrated in Figure 1.15.

\textbf{Example 17:}

\[
x(t) = e^{-at}u(t)
\]

\[
h(t) = u(t)
\]

Evaluate \( y(t) = x(t) * h(t) \). Referring to Figure 1.16 the convolution, \( y(t) \), of \( x(t) \) and \( h(t) \) is found.

(a) \( t < 0 \); \( y(t) = 0 \)

(b) \( t > 0 \);

\[
y(t) = \int_{0}^{t} x(\tau)h(t-\tau)d\tau = \int_{0}^{t} e^{-a\tau}d\tau
\]

\[
= \frac{1}{a}(1 - e^{-at}), \quad t > 0
\]

\[
= \frac{1}{a}(1 - e^{-at})u(t)
\]
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\[ x(\tau) \]

\[ h(\tau) \]

\[ h(-\tau) \]

\[ h(t-\tau) \]

Figure 1.15: The Convolution process

Example 18: Evaluate \( y(t) = x(t) * h(t) \)

\[
x(t) = \begin{cases} 
1, & 0 < t < T \\
0, & \text{otherwise}
\end{cases} \quad (1.24)
\]

\[
h(t) = \begin{cases} 
t, & 0 < t < 2T \\
0, & \text{otherwise}
\end{cases} \quad (1.25)
\]

Step 1: Draw \( x(\tau) \) and \( h(\tau) \).
Step 2: Flip \( h(\tau) \) (or \( x(\tau) \)).
Step 3:
1. Shift \( h(-\tau) \) (or \( x(-\tau) \))
2. Multiply by \( x(\tau) \)
3. Integrate over region of overlap.

a. \( t \leq 0 \), no overlap

\[ y(t) = 0 \]

b. \( 0 < t \leq T \)
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Step 1:
- draw \( x(\tau) \)
- and \( h(\tau) \)

Step 2:
- get \( h(-\tau) \)

Step 3:
- shift \( h(-\tau) \)
- multiply by \( x(\tau) \)
- integrate

Output result from
- \( x(\tau) \ast h(\tau) \)

Figure 1.16: Illustration of Example 17
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\[
h(-\tau) = \begin{cases} 
2T & \text{for } -2T \leq \tau < 0 \\
0 & \text{for } 0 \leq \tau < 2T 
\end{cases}
\]

The equation of this line is \( h(t - \tau) = t - \tau \).

\[
y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau \\
= \int_{0}^{t} (t - \tau) d\tau \\
= \frac{1}{2} t^2; \quad 0 < t < T \tag{1.26}
\]

c. \( T < t \leq 2T \)

\[
y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau \\
= \int_{0}^{T} (t - \tau) d\tau \\
= Tt - \frac{1}{2} T^2; \quad T < t < 2T \tag{1.27}
\]

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d. \(2T < t \leq 3T\)

\[
y(t) = \int_{-2T+t}^{T} 1(t-\tau) d\tau = -\frac{1}{2} t^2 + T t + \frac{3}{2} T^2; \quad 2T < t < 3T
\]

(1.28)

e. \(t > 3T\)
1.2. ONE-DIMENSIONAL CONTINUOUS-TIME SYSTEMS

\[ y(t) = \int_{-2T+t}^{T} 1(t-\tau)d\tau = 0; \quad t > 3T \]

Hence,
\[
y(t) = \begin{cases}
0, & t \leq 0 \\
\frac{1}{2}t^2, & 0 < t \leq T \\
Tt - \frac{1}{2}T^2, & T < t \leq 2T \\
-\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2, & 2T < t \leq 3T \\
0, & t > 3T
\end{cases}
\]

The summary of the steps in Example 18 are illustrated in Figure 1.17.

More Properties for LTI Systems

A. Causality for LTI Systems

Recall, a system is said to be causal (or non-anticipative) if its output at time \( t \) does not depend on the input after time \( t \); it depends only on the input applied before and/or at time \( t \). In short, the current output does not depend on future inputs. Now for LTI System with impulse response \( h(t) \), we have
\[
y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \tag{1.29}
\]
Suppose the input \( x(\cdot) \) is applied at \( t = \tau \); that is,
\[
x(\tau) \neq 0 \tag{1.30}
\]
Assume zero initial conditions, then
\[
y(t) = 0 \quad \text{for } t < \tau \tag{1.31}
\]
for causal system. From 1.31, the integral in 1.29 will have the following form
\[
y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau \tag{1.32}
\]
\[
\equiv \int_{t}^{\infty} h(\tau)x(t-\tau)d\tau \tag{1.33}
\]
Suppose \( \tau = 0 \); i.e., \( x(0) \neq 0 \) in 1.30, hence \( y(t) = 0 \) for \( t < 0 \) in 1.31, hence \( h(t-\tau) \equiv h(t) \) must be zero for the integral in 1.32 to be zero when \( \tau = 0 \). Hence, for causality of a LTI System,
\[
h(t) = 0; \quad \text{for } t < 0 \tag{1.34}
\]
Figure 1.17: Illustration of the steps used to obtain $y(t) = x(t) \ast h(t)$ in Example 18
1.2. ONE-DIMENSIONAL CONTINUOUS-TIME SYSTEMS

B. Stability of L.T.I. System

For a LTI System,

\[ y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \]  

(1.35)

Now, a system is bounded-input-bounded-output stable (BIBO Stable) if for every input \( x(t) \), which is bounded, (i.e. \( |x(t)| < B \) for all \( t \)) the output is bounded.

Assuming \( x(t) \) is bounded; i.e. \( |x(t)| < B \),

\[
|y(t)| = |\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau| \\
\leq \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)|d\tau \\
\leq B \int_{-\infty}^{\infty} |h(\tau)|d\tau
\]

(1.36)

will be bounded if \( h(t) \) is absolutely integrable. Now, assume \( h(t) \) is bounded; i.e. \( |h(t)| < K \)

\[
|y(t)| = |\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau| \\
\leq K \int_{-\infty}^{\infty} |x(t-\tau)|d\tau
\]

(1.37)

Hence, bounded inputs will provide bounded outputs. Therefore, we have the following theorem for the stability of LTI Systems.

**Theorem:** A LTI System is BIBO stable if and only if

\[ \int_{-\infty}^{\infty} |h(t)|dt < \infty \]  

(1.38)

C. The Step Response for LTI Systems

Again, for LTI System,

\[ y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \]  

(1.39)

Let \( x(t) = u(t) \); a unit step function.

\[ y(t) = \int_{-\infty}^{t} h(\tau)d\tau \]  

(1.40)
Differentiate 1.40 with respect to \( t \) (using Libnitz rule), we get

\[ h(t) = \frac{d}{dt} y(t) \]  \hspace{1cm} (1.41)

That is, the impulse response \( h(t) \) for LTI System is the derivative of the step response.

**Exercise**:

1. Is the system with input and output as shown causal? Why?

2. If \( x_1(t) \), \( x_2(t) \), and \( h(t) \) are arbitrary signals, and \( \alpha \) is a constant, show the following:
   
   a. \( h(t) \ast [x_1(t) + x_2(t)] = h(t) \ast x_1(t) + h(t) \ast x_2(t) \)
   
   b. \( h(t) \ast [x_1(t) \ast x_2(t)] = (h(t) \ast x_1(t)) \ast x_2(t) \)
   
   c. \( h(t) \ast [\alpha x_1(t)] = \alpha h(t) \ast x_1(t) \).

3. The following system has an impulse response \( h(t) = \frac{1}{RC} e^{-t/RC} u(t) \).

   Is the system causal? Is it BIBO stable?

4. Find and sketch \( y(t) \) using \( x(t) \) and \( h(t) \).

   \[ x(t) = 2e^{-10t} u(t), \]
   
   \[ h(t) = u(t - 2), \]


1.3 Signal Representation

As we pointed out before, a number of signal processing practices can be modeled as an input-output systems relations. This is particularly common in image enhancement, restoration, and coding. If the system is L.T.I., applications such as filtering, transmission, etc. can be better understood in the frequency domain, instead of the time (or spatial) domain. Frequency domain representation is obtained by the Fourier Series expansion for periodic signals (or images), and by the Fourier transform for a periodic signals (or images). We are interested mainly in the Fourier transforms, but first we will recall the Fourier Series definition.

1.3.a Fourier Series

The Fourier Series is a representation of periodic signals using trigonometric functions. Basically, any periodic signal \( x(t) \) can be expressed in terms of trigonometric functions in the following form:

\[
x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t \tag{1.42}
\]
Where

\[ \omega_0 = \frac{2\pi}{T} \]

\[ T \text{ = period} \]

\[ a_n = \frac{2}{T} \int_T x(t) \cos(n\omega_0 t) dt \quad n=1,2,\ldots,\infty \] (1.44)

\[ b_n = \frac{2}{T} \int_T x(t) \sin(n\omega_0 t) dt \quad n=1,2,\ldots,\infty \]

\[ a_0 = \frac{1}{T} \int_T x(t) dt \] (1.45)

\( a_0 \) is called the D-C L term, \( \{a_n\} \) are the even coefficients, and \( \{b_n\} \) are the odd coefficients.

**Example 19:**

Find the Fourier Series expansion of the square wave defined by the periodic extension of

\[ x(t) = \begin{cases} A & 0 < t < T/2 \\ -A & T/2 < t < T \end{cases} \] (1.46)

\[ x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t) \] (1.47)
1.3. SIGNAL REPRESENTATION

\[ a_o = \frac{1}{T} \int_T x(t) dt \quad \text{(i.e., the average)} \]
\[ \equiv 0 \]

\[ a_n = \frac{2}{T} \int_T x(t) \cos n\omega_o t dt, \quad n=1,2,... \]
\[ = \frac{2}{T} \left[ \int_0^{T/2} A \cos n\omega_o t dt + \int_{T/2}^T -A \cos n\omega_o t dt \right] \]
\[ \equiv 0 \]

\[ b_n = \frac{2}{T} \int_T x(t) \sin n\omega_o t dt, n=1,2,... \]
\[ b_n = \frac{2}{T} \left[ \int_0^{T/2} A \sin n\omega_o t dt + \int_{T/2}^T -A \sin n\omega_o t dt \right] \]
\[ = \frac{2A}{T} \left[ \frac{\cos n\omega_o t}{n\omega_o} \bigg|_{T/2}^{T} + \frac{\cos n\omega_o t}{n\omega_o} \bigg|_{T/2}^{0} \right] \]

Let

\[ \omega_o = \frac{2\pi}{T} \quad \text{(which is the definition!)} \]

\[ b_n = \frac{2A}{n\pi} (1 - \cos n\pi) \]
\[ = \begin{cases} 
\frac{4A}{n\pi}, & n \text{ odd} \\
0, & n \text{ even} 
\end{cases} \]

Hence,

\[ x(t) = \frac{4A}{\pi} \left[ \sin \omega_o t + \frac{1}{3} \sin 3\omega_o t + \frac{1}{5} \sin 5\omega_o t + ... \right]. \quad (1.48) \]

Facts

1. If \( x(t) \) is odd \( a_n = 0, \ n=0,1,2,... \)
2. If \( x(t) \) is even \( b_n = 0, \ n=1,2,... \)

Another from of the Fourier series; expansion can be easily shown to be:

\[ a_n \cos n\omega_o t + b_n \sin \omega_o t = A_n \cos(n\omega_o t + \theta_n) \quad (1.49) \]
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\[ x(t) = x(-t) \]  
\[ x(t) = -x(-t) \]

Figure 1.18: Example of an even and odd function

Where

\[ A_n = \sqrt{a_n^2 + b_n^2} \]  
\[ \theta_n = -\tan^{-1}\left(\frac{b_n}{a_n}\right) \]

Therefore, 1.55 can be rewritten as

\[ x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \theta_n), \]  
\[ A_0 = a_0 = \frac{1}{T} \int_T x(t) dt, \]
\[ A_n = \sqrt{a_n^2 + b_n^2}, \]
\[ a_n = \frac{1}{T} \int_T x(t) \cos n\omega_0 t dt, \]
\[ b_n = \frac{1}{T} \int_T x(t) \sin n\omega_0 t dt. \]

Figure 1.19 lists several coefficients for the complex exponential Fourier series of several signals

A third form of the Fourier series expansion is using exponential functions instead of trigonometric functions. The periodic signal can be expressed in this alternative form rather than 1.30. This is known as the complex exponential form,

\[ x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}, \]  
\[ X_n = \frac{1}{T} \int_T x(t) e^{-jn\omega_0 t} dt. \]
1.3. SIGNAL REPRESENTATION

1. Half-rectified sine wave
\[ X_n = \begin{cases} \frac{A}{\pi(1-n^2)} & n = 0, \pm 2, \pm 4 \\ 0, & n \text{ odd and } \neq 1 \\ -\frac{1}{4}jnA & n = \pm 1 \end{cases} \]

2. Full-rectified sine wave
\[ X_n = \begin{cases} \frac{2A}{\pi(1-n^2)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \]

3. Pulse-train signal
\[ X_n = \frac{A_T}{\pi} \text{sinc} n f_o \tau e^{-j2\pi n f_o t_o} \quad f_o = T_o^{-1} \]

4. Square wave
\[ X_n = \begin{cases} \frac{2A}{10\pi} & n = \pm 1, \pm 5, \ldots \\ \frac{-2A}{10\pi} & n = \pm 3, \pm 7, \ldots \\ 0 & n \text{ even} \end{cases} \]

5. Triangular wave
\[ X_n = \begin{cases} \frac{4A}{\pi n^2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \]

Figure 1.19: Coefficients for the Complex Exponential Fourier Series of several signals

Recall, if \( x(t) \) is periodic with period \( T \), then from 1.30

\[ x(t) = a_o + \sum_{n=1}^{\infty} a_n \cos n\omega_o t + \sum_{n=1}^{\infty} b_n \sin n\omega_o t \] (1.55)

Notes
1. The Coefficients \( X_n \) in 1.54 are, in general, complex; i.e.,
\[ X_n = \text{Re}(X_n) + j\text{Im}(X_n) \] (1.56)
\[ j = \sqrt{-1} \]

2. The quantity

\[ |X_n| \triangleq \sqrt{|\text{Re}(X_n)|^2 + |\text{Im}(X_n)|^2}, \]  

is called the magnitude of the line spectrum of the periodic signal \( x(t) \).

3. The quantity

\[ \text{ang} (X_n) \triangleq \tan^{-1} \left[ \frac{\text{Im}(X_n)}{\text{Re}(X_n)} \right], \]  

is called the phase spectrum of \( x(t) \).

4. \( A_n \) and \( \theta_n \) in 1.50-1.51 are related to \( X_n \) by

\[ A_n = 2 |X_n| \]  
\[ \theta_n = \text{ang} X_n \]  

5. Figure 1.20 summarizes the three forms of the Fourier Series.

**Example 20:**

\( x(t) \) is a pulse train.

Pulse width = \( \tau \)

Period \( T = T_0 \)

\[
X_n = \frac{1}{T_0} \int_{t_0}^{t_0+\tau} x(t)e^{-jn\omega_0 t} dt
\]

\[
= \frac{1}{T_0} \left[ Ae^{-jn\omega_0 t_0} + \int_{t_0+\tau/2}^{t_0-\tau/2} Ae^{-jn\omega_0 t} dt \right]
\]

\[
= \frac{2A}{\omega_0 T_0} e^{-jn\omega_0 t_0} \sin \left( \frac{\omega_0 \tau}{2} \right), \quad n \neq 0
\]

Substitute

\[ \omega_0 = \frac{2\pi}{T_0} \triangleq 2\pi f_0, \quad f_0 \triangleq \frac{1}{T}, \]  

\[
X_n = \frac{A\tau \sin \pi n f_0 \tau}{T_0} e^{-j2\pi n f_0 t_0}
\]

\[ \triangleq |X_n| e^{\text{ang} X_n} \]  

Line Spectrum,
### 1.3. SIGNAL REPRESENTATION

<table>
<thead>
<tr>
<th>Type</th>
<th>Formula</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Trigonometric sine-cosine</td>
<td>$x(t) = a_o + \sum_{n=1}^{\infty} (a_n \cos n\omega_o t + b_n \sin n\omega_o t)$</td>
<td>$a_o = \frac{1}{T_o} \int_{T_o}^{T_o} x(t) dt$&lt;br&gt;$a_n = \frac{2}{T_o} \int_{T_o}^{T_o} x(t) \cos n\omega_o t dt$&lt;br&gt;$b_n = \frac{2}{T_o} \int_{T_o}^{T_o} x(t) \sin n\omega_o t dt$</td>
</tr>
<tr>
<td>2. Trigonometric cosine</td>
<td>$x(t) = A_o + \sum_{n=1}^{\infty} A_n \cos(n\omega_o t + \theta_n)$</td>
<td>$A_o = a_o$&lt;br&gt;$A_n = \sqrt{a_n^2 + b_n^2}$&lt;br&gt;$\theta_n = -\tan^{-1}\frac{b_n}{a_n}$</td>
</tr>
<tr>
<td>3. Complex exponential</td>
<td>$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j n \omega_o t}$</td>
<td>$X_n = \frac{1}{T_o} \int_{T_o}^{T_o} x(t) e^{-j n \omega_o t} dt$&lt;br&gt;$X_n = \left{ \begin{array}{ll} \frac{1}{2}(a_n - j b_n) &amp; n &gt; 0 \ \frac{1}{2}(a_n - j b_n) &amp; n &lt; 0 \ X_n^* &amp; n = 0, n \text{ even}, for x(t) real \end{array} \right.$</td>
</tr>
</tbody>
</table>

---

Figure 1.20: Summary of Fourier Series Properties

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Figure 1.21: Example of a pulse train

\[ |X_n| = \frac{A\tau}{T_0} \text{sinc}(nf_0\tau) \]  
\[ \text{ang } X_n = -2\pi nf_0t_o \]
Exercise:

Find the Fourier Series expansions for the line spectrum waveforms. Plot the line spectrum for each waveform from the exponential Fourier Series representation. Assume all amplitudes to be equal to one.

1.3. SIGNAL REPRESENTATION

1.3.b The Fourier Transform

Definition

The Fourier transform of a function \( x(t) \), \( X(f) \), is defined as

\[
X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt. \quad (\diamond)
\]

The inverse Fourier transform of a function \( X(f) \), \( x(t) \), is defined as

\[
x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df. \quad (1.66)
\]

The integrals in 1.65 and 1.66 define what’s called a Fourier Transform Pair; symbolically, we write

\[
x(t) \longleftrightarrow X(f) \quad (1.67)
\]

to denote that \( x(t) \) and \( X(f) \) are Fourier transform pairs.

Existence of the Fourier Transform

(i) If

\[
\int_{-\infty}^{\infty} |x(t)| \, dt < \infty, \quad (i.e., x(t) \in L_1)
\]

then

a) \( X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \) exists, \ i.e., \( |X(f)| < \infty \)

b) \( X(f) \) is continuous

c) \( X(f) \to 0 \) as \( f \to \pm \infty \)

(Riemann - Lebesgue Lemma)
(ii) In some cases, the condition in 1.68 might not hold, but we still can define the Fourier transform. For example, the unit step function \( u(t) \), the sinusoidal functions \( \sin at \) or \( \cos bt \), some periodic functions (besides the sinusoids)... etc. do not satisfy 1.68, yet their Fourier transform exists and is well defined. Also, some functions which do not exist physically, e.g., the delta function, functions of infinite discontinuities (e.g., \( \sin \frac{1}{t} \)), etc. also have Fourier Transforms.

**Fact:**

If \( x(t) \in L_1 \) and \( X(f) \in L_1 \), i.e., \( \int_{-\infty}^{\infty} |x(t)| \, dt < \infty \) and \( \int_{-\infty}^{\infty} |X(f)| \, df < \infty \), then the pair \( x(t) \leftrightarrow X(f) \) is unique; i.e., no two different functions can have the same Fourier transform or the same inverse Fourier transform.

**Examples**

**Example 21:**

\[
x(t) = \begin{cases} 
1, & \text{if } |t| < \frac{a}{2} \\
0, & \text{otherwise} 
\end{cases}
\]  \hspace{1cm} (1.69)

\[x(t)\triangleq P_{\frac{a}{2}}(t) \triangleq \Pi(t/a) \triangleq \text{rect}(t/a)\]
1.3. SIGNAL REPRESENTATION

\[ X(f) \triangleq \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \]

\[ = \int_{-\alpha}^{\alpha} e^{-j2\pi ft} dt \]

\[ = \frac{1}{j2\pi f} e^{-j2\pi ft} \bigg|_{-\alpha}^{\alpha} \]

\[ = e^{-j2\pi fa/2} - e^{j2\pi fa/2} \]

\[ = -j2\pi f \]

\[ = \frac{1}{\pi f} \left( e^{j\pi fa} - e^{-j\pi fa} \right) \]

\[ \triangleq \frac{1}{\pi f} \sin(\pi fa) \]

\[ \equiv a\sin \pi fa \]

\[ \equiv a \text{sinc}(af), \quad (1.70) \]

where

\[ \text{sinc } \alpha \triangleq \frac{\sin \pi \alpha}{\pi \alpha} \quad (1.71) \]

Notes

a. In general \( X(f) \) is a complex quantity.

b. Later on we will prove that if \( x(t) \) is even function of \( t \), then \( X(f) \) is real and even function of \( f \).

c. Plot of

\[ \text{sinct} \triangleq \frac{\sin \pi t}{\pi t} \quad (1.72) \]

Zero crossings:

\[ \sin \pi t = 0 \]

\[ \pi t = \pm n\pi \]

\[ t = \pm n = \pm 1, \pm 2, ... \]
at \( t=0 \)

\[
\lim_{t \to 0} \frac{\sin \pi t}{\pi t} = \lim_{t \to 0} \pi \cos \pi t = 1 \tag{1.73}
\]

\[
sinc \ t = 1 \tag{1.74}
\]

**Example 22:**

\[
x(t) = e^{-\alpha t}u(t), \quad \alpha > 0 \tag{1.75}
\]

Figure 1.23: Plot of \( e^{-\alpha t}u(t) \)

\[
X(f) \triangleq \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt
\]

\[
= \int_{0}^{\infty} e^{-\alpha t}e^{-j2\pi ft}dt
\]

\[
= \int_{0}^{\infty} e^{-(\alpha+j2\pi f)t}dt
\]

\[
= \frac{-1}{(\alpha + j2\pi f)}e^{-(\alpha+j2\pi f)t} \bigg|_{0}^{\infty}
\]

\[
= \frac{1}{\alpha + j2\pi f} \tag{1.76}
\]

\[
= \frac{\alpha - j2\pi f}{(\alpha + j2\pi f)(\alpha - j2\pi f)}
\]

\[
= \frac{\alpha}{\alpha^2 + (2\pi f)^2} - \frac{j}{\alpha^2 + (2\pi f)^2} \tag{1.77}
\]

\[
X(f) = X_r(f) + jX_i(f), \tag{1.78}
\]

Where \( X_r(f) \) is the real part and \( X_i(f) \) is the imaginary part. The plot of \( X(f) \) will be in terms of \( X_r(f) \) and \( X_i(f) \) or \( |X(f)| \) and \( \text{ang } X(f) \).
1.3. **SIGNAL REPRESENTATION**

\[ |X(f)| \triangleq \sqrt{X_r^2(f) + X_i^2(f)} \]
\[ = \frac{1}{\sqrt{\alpha^2 + (2\pi f)^2}} \]  \hspace{1cm} (1.79)

ang \( X(f) \) = \[ \tan^{-1} \frac{X_i(f)}{X_r(f)} \]
\[ = \tan^{-1} \left( -\frac{2\pi f}{\alpha} \right) \]  \hspace{1cm} (1.80)

![Diagram](image)

Figure 1.24: Plot of \( X(f) \) in Example 22

---

**Example 23:**

\[ x(t) = e^{-\alpha|t|} \]  \hspace{1cm} (1.81)

(even)
\[ X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \]
\[ = \int_{-\infty}^{0} e^{\alpha t} e^{-j2\pi ft} dt + \int_{0}^{\infty} e^{-\alpha t} e^{-j2\pi ft} dt \]
\[ = \int_{-\infty}^{0} e^{(\alpha-j2\pi f)t} dt + \int_{0}^{\infty} e^{-(\alpha+j2\pi f)t} dt \]
\[ = \frac{1}{\alpha - j2\pi f} e^{(\alpha-j2\pi f)t} \bigg|_{0}^{\infty} + \frac{1}{\alpha + j2\pi f} e^{-(\alpha+j2\pi f)t} \bigg|_{0}^{\infty} \]
\[ = \frac{1}{\alpha - j2\pi f} + \frac{1}{\alpha + j2\pi f} \]
\[ = \frac{2\alpha}{\alpha^2 + (2\pi f)^2} \]  
(1.82) 

Properties of the Continuous-time Fourier Transform

(i) **LINEARITY**

If \( x_1(t) \longleftrightarrow X_1(f) \)

\( x_2(t) \longleftrightarrow X_2(f) \), then

\( x_3(t) = ax_1(t) + bx_2(t) \longleftrightarrow aX_1(f) + bX_2(f) \)  
(1.83)

**Proof:**
By definition, the Fourier Transform of $x_3(t)$ is

$$
\begin{align*}
\mathcal{F}\{x_3(t)\} &= \int_{-\infty}^{\infty} (ax_1(t) + bx_2(t)) e^{-j2\pi ft} dt \\
&= a \int_{-\infty}^{\infty} x_1(t)e^{-j2\pi ft} dt + b \int_{-\infty}^{\infty} x_2(t)e^{-j2\pi ft} dt \\
&\triangleq aX_1(f) + bX_2(f)
\end{align*}
$$

(1.84)

Example 24:

$$
\begin{align*}
x(t) &= 3\Pi \left( \frac{t}{a} \right) + 15e^{-4t}u(t) \\
X(f) &= 3\mathcal{F}(\Pi(t/a)) + 15\mathcal{F}(e^{-4t}u(t)) \\
&= 3\text{sinc}(af) + 15 \frac{1}{4 + j2\pi f} \tag{1.85}
\end{align*}
$$

from 1.70

$$
\begin{align*}
\text{from 1.76}
\end{align*}
$$

(ii) SYMMETRY PROPERTIES

Let $x(t)$ be a real-valued function of time, then

$$
X(-f) = \overline{X(f)}, \tag{1.86}
$$

Where the bar denotes complex conjugate.

Proof:

By definition:

$$
X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} dt \tag{1.87}
$$

Taking the conjugate of the two sides of 1.87
\[
X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt
\]

\[
= \int_{-\infty}^{\infty} x(t)e^{j2\pi ft}dt
\]

\[
\equiv \int_{-\infty}^{\infty} x(t)e^{j2\pi ft}dt, \quad x(t) \text{ real}
\]

\[
\triangleq X(-f)
\]

\[(1.88)\]

\[(1.89)\]

\[\text{Q.E.D.}\]

**Example 25:**

\[
x(t) = e^{-\alpha t}u(t)
\]

\[
X(f) = \frac{1}{\alpha + j2\pi f}
\]

\[
X(-f) = \frac{1}{\alpha - j2\pi f}
\]

\[
X(f) = \frac{1}{\alpha - j2\pi f}
\]

\[\diamondsuit \diamondsuit \diamondsuit\]

Any function can be unambiguously expressed in terms of an *even* part plus an *odd* part; that is, for a general function \(x(t)\), we can decompose it into

\[
x(t) = x_e(t) + x_o(t),
\]

(1.90)

Where

\(x_e(t)\) is the even part of \(x(t)\) and

\(x_o\) is the odd part of \(x(t)\).

As we stated earlier, \(X(f)\) is, in general, complex; i.e.,

\[
X(f) = X_r(f) + jX_i(f)
\]

(1.91)

**Corollary 1:** If \(x(t)\) is real, then \(X_r(f)\) is even and \(X_i(f)\) is odd.
1.3. SIGNAL REPRESENTATION

Proof:

\[ X(f) = X_r(f) + jX_i(f) \]
\[ X(f) = X_r(f) - jX_i(f) \] (1.92)

Also,

\[ X(-f) = X_r(-f) + jX_i(-f) \] (1.93)

But \( X(f) = X(-f) \), from 1.86. Hence, by equating the real and the imaginary parts in 1.92 and 1.93, we get

\[ X_r(f) = X_r(-f); \quad \text{i.e., even} \] (1.94)
\[ X_i(f) = -X_i(-f); \quad \text{i.e., odd} \] (1.95)

Q.E.D.

Corollary 2: If \( x(t) \) is real and even, then \( X(f) \) is real and even.

Proof: Given,

\[ x(t) = x(-t) \] (1.96)

Now,

\[ X(f) \triangleq \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} dt \] (1.97)
\[ \equiv \int_{-\infty}^{\infty} x(-t)e^{-j2\pi ft} dt \] (1.98)

Let

\[ \alpha = -t \]
\[ d\alpha = -dt \]

where \( t \) varies from \(-\infty \) to \( \infty \), and \( \alpha \) varies from \(-\infty \) to \( \infty \), therefore 1.98 becomes
\[ \begin{align*}
&= \int_{-\infty}^{\infty} x(\alpha)e^{j2\pi f\alpha} d\alpha \\
\equiv & \int_{-\infty}^{\infty} x(\alpha)e^{-j2\pi(-f)\alpha} d\alpha \\
\triangleq & X(-f) 
\end{align*} \]

From 1.97 and 1.99

\[ X(f) = X(-f); \text{ i.e., even} \] (1.100)

Now, is it real?

\[ \text{i.e., } \overline{X(f)} \stackrel{?}{=} X(f) \] (1.101)

Well,

\[ \overline{X(f)} = X(-f) \text{ from 1.86.} \] (1.102)

Hence,

\[ \overline{X(f)} = X(f) \text{ from 1.100.} \] (1.103)

Therefore, if \( x(t) \) is real and even, \( X(f) \) will be real and even too!

Q.E.D.

Example 26:

\[ x(t) = e^{-\alpha|t|} \] (1.104)

\[ X(f) = \frac{2\alpha}{\alpha^2 + (2\pi f)^2} \] (1.105)

Exercises

1. Show that if \( x(t) \) is real and odd then \( X(f) \) is imaginary and odd
2. Show that if $x(t)$ is imaginary and even, then $X(f)$ is also imaginary and even.

(iii) **Time Shifting**

Consider the Fourier Transform pair $x(t) \longleftrightarrow X(f)$, then $x(t - t_0) \longleftrightarrow e^{-j2\pi ft_0}X(f)$.

**Proof:**

$$
\mathcal{F}(x(t - t_0)) \triangleq \int_{-\infty}^{\infty} x(t - t_0)e^{-j2\pi ft}dt (1.106)
$$

Let

$$
t - t_0 = \alpha \\
t = \alpha + t_0 \\
dt = d\alpha
$$

where $t$ varies from $-\infty$ to $\infty$, and $\alpha$ varies from $-\infty$ to $\infty$. Hence,

$$
\mathcal{F}(x(\alpha)) = \int_{-\infty}^{\infty} x(\alpha)e^{-j2\pi f(\alpha + t_0)}d\alpha (1.107)
$$

$$
= e^{-j2\pi ft_0} \int_{-\infty}^{\infty} x(\alpha)e^{-j2\pi f\alpha}d\alpha \\
\triangleq e^{-j2\pi ft_0}X(f)
$$

**Q.E.D.**

**Example 27:**

We know that $\Pi(t/a) \longleftrightarrow a \text{sinc}(fa)$ from 1.70.

Hence,

$$
\Pi \left( \frac{t - \alpha}{a} \right) \longleftrightarrow e^{-j2\pi f\alpha} a \text{sinc}(fa) (1.108)
$$
(iv) Differentiation

If \( x(t) \leftrightarrow X(f) \), then

\[
\frac{d}{dt} x(t) \leftrightarrow j2\pi f X(f)
\]  
(1.109)

**Proof:**

\[
x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft} df
\]  
(1.110)

Differentiating both sides with respect to \( t \),

\[
\frac{d}{dt} x(t) = \frac{d}{dt} \int_{-\infty}^{\infty} X(f)e^{j2\pi ft} df
\]

\[
= \int_{-\infty}^{\infty} X(f) \left( \frac{d}{dt}e^{j2\pi ft} \right) df
\]

\[
= \int_{-\infty}^{\infty} X(f)j2\pi fe^{j2\pi ft} df
\]

\[
\Delta = \int_{-\infty}^{\infty} X_1(f)e^{j2\pi ft} df
\]  
(1.111)

Comparing 1.110 and 1.111, hence,

\[
\frac{d}{dt} x(t) \leftrightarrow X_1(f) = j2\pi f X(f)
\]  
(1.112)

**Example 28:**

\[
e^{-\alpha t}u(t) \leftrightarrow \frac{1}{\alpha + j2\pi f}
\]

\[
e^{-\alpha t}u'(t) - \alpha e^{-\alpha t}u(t) \leftrightarrow j2\pi f \frac{1}{\alpha + j2\pi f}
\]  
(1.113)

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1.3. **SIGNAL REPRESENTATION**

Generalization: The result in 1.109 can be generalized to the $n^{th}$ derivative. It is trivial to show that:

\[ x(t) \leftrightarrow X(f), \quad \text{then} \]
\[ \frac{d^n}{dt^n} x(t) \leftrightarrow (j2\pi f)^n X(f) \quad (1.115) \]

(v) **TIME AND FREQUENCY SCALING**

If 
\[ x(t) \leftrightarrow X(f), \quad \text{then,} \]
\[ x(at) \leftrightarrow \frac{1}{|a|} X \left( \frac{f}{a} \right) \]

Where $a$ is a real constant.

**Proof:**

a. Let $a > 0$

\[ \mathcal{F}(x(at)) = \int_{-\infty}^{\infty} x(at) e^{-j2\pi ft} dt \quad (1.116) \]

Let,

\[ \alpha = at, \]
\[ t = \frac{\alpha}{a} \]
\[ dt = \frac{1}{a} d\alpha, \]

where $t$ varies from $-\infty$ to $\infty$, and $\alpha$ varies from $-\infty$ to $\infty$. Hence,

\[ \mathcal{F}(x(at)) = \frac{1}{a} \int_{-\infty}^{\infty} x(\alpha) e^{-j2\pi f\alpha/a} d\alpha \]
\[ = \frac{1}{a} \int_{-\infty}^{\infty} x(\alpha) e^{-j2\pi(f/a)\alpha} d\alpha \]
\[ = \frac{1}{a} X(f/a) \quad (1.117) \]
b. Let \( a < 0 \), i.e.,\[ a = -b, \quad b = |a| \quad (1.118) \]

\[
\mathcal{F}(x(at)) = \int_{-\infty}^{\infty} x(at) e^{-j2\pi ft} dt
= \int_{-\infty}^{\infty} x(-bt) e^{-j2\pi ft} dt
\]

Let
\[
\beta = -bt, \quad t = -\frac{\beta}{b}, \quad d\beta = -b dt,
\]

where \( t \) varies from \(-\infty\) to \( \infty \), and \( \beta \) varies from \(-\infty\) to \( \infty \). Hence,

\[
\mathcal{F}(x(at)) = \left(-\frac{1}{b}\right) \int_{-\infty}^{\infty} \frac{1}{|a|} x(\beta) e^{j2\pi(f/b)\beta} d\beta
= \frac{1}{|a|} \int_{-\infty}^{\infty} x(\beta) e^{-j2\pi(-f/b)\beta} d\beta
= \frac{1}{|a|} \int_{-\infty}^{\infty} x(\beta) e^{-j2\pi(f/a)\beta} d\beta
= \frac{1}{|a|} X(f/a)
\quad (1.119)
\]

From 1.117 and 1.118, if

\[ x(t) \leftrightarrow X(f), \quad \text{then} \]
\[ x(at) \leftrightarrow \frac{1}{|a|} X(f/a) \]

Q.E.D.
1.3. SIGNAL REPRESENTATION

Example 29:

Hence, the time resolution (e.g., the pulse width) is inversely proportional to the frequency resolution (e.g., the main lobe width). This is known as the uncertainty principle.

(vi) DUALITY

If \( x(t) \leftrightarrow X(f) \), then
\[
X(t) \leftrightarrow x(-f)
\] (1.120)

Proof:

\[
X(u) = \int_{-\infty}^{\infty} x(v)e^{-j2\pi uv}dv
\] (1.121)

is the Fourier Transform of \( x(\cdot) \); we simply used the parameter \( v \) instead of \( t \) and the parameter \( u \) instead of \( f \).
Now, let \( u = f \) and \( v = t \) in 1.121

\[
X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt
\]

(1.122)

On the other hand, suppose we put \( u = t \) and \( v = f \) in 1.121, then

\[
X(t) = \int_{-\infty}^{\infty} x(f) e^{-j2\pi tf} df
\]

\[
\equiv \int_{-\infty}^{\infty} x(f) e^{j2\pi t(-f)} df
\]

Therefore,

\[
X(t) \longleftrightarrow x(-f)
\]

(1.123)

(1.124)

Q.E.D.

The implications of duality can be illustrated in the following example.

**Example 30:**

\[
\Pi\left(\frac{t}{2T}\right) \triangleq \begin{cases} 
1, & |t| < T \\
0, & \text{otherwise}
\end{cases}
\]

(1.125)

We know that

\[
\Pi(t/2T) \longleftrightarrow 2T \text{sinc} 2Tf;
\]

(1.126)

This is illustrated in Figure 1.26.

Now, suppose we want to obtain the Fourier Transform of

\[
x(t) = 2T \text{sinc} 2Tf
\]

(1.127)

We can use the definitions and integrate, or, much better, we can use the duality principle.

\[
2T \text{sinc} 2Tf \longleftrightarrow \Pi(t/2T) \mid_{t=-f} = \Pi(-f/2T)
\]

(1.128)
1.3. SIGNAL REPRESENTATION

Figure 1.26: Fourier Transform of a pulse

Figure 1.27: Fourier Transform of a sinc

This is shown in Figure 1.27.

Example 31:

\[ x(t) = e^{-\alpha|t|} \]
\[ X(f) = \frac{2\alpha}{\alpha^2 + (2\pi f)^2} \]

Now suppose we want to find the Fourier Transform of

\[ y(t) = \frac{2\alpha}{\alpha^2 + (2\pi t)^2} \]

From duality, it is evident that
\[ Y(f) = e^{-\alpha|f|} = e^{-\alpha|f|} \]  \hspace{1cm} (1.130)

Graphically, this is shown in Figure 1.28.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Illustration of the Duality Property}
\end{figure}

(vii) **The Parseval’s Theorem**

\[ \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df, \]  \hspace{1cm} (1.131)

Parseval’s Theorem says that the energy of a certain signal is the same in the time and the frequency domains.

**Proof:**
\[ \int_{-\infty}^{\infty} |x(t)|^2 \, dt \triangleq \int_{-\infty}^{\infty} x(t)\overline{x(t)} \, dt \]
\[ = \int_{-\infty}^{\infty} \overline{x(t)} \left( \int_{-\infty}^{\infty} X(f)e^{j2\pi ft} \, df \right) \, dt \]
\[ = \int_{-\infty}^{\infty} X(f) \left( \int_{-\infty}^{\infty} \overline{x(t)}e^{j2\pi ft} \, dt \right) \, df \]
\[ = \int_{-\infty}^{\infty} X(f) \left( \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} \, dt \right) \, df \]
\[ = \int_{-\infty}^{\infty} X(f)\overline{X(f)} \, df \]
\[ \triangleq \int_{-\infty}^{\infty} |X(f)|^2 \, df \] \hfill (1.132)

Note \( \overline{A B} = \overline{A} \overline{B} \).

The above theorem says that the total energy can be calculated by computing the energy per unit time \( |x(t)|^2 \) and integrating over time or by computing the energy per unit frequency \( |X(f)|^2 \) and integrating over all frequencies.

**Example 32:**

Evaluate the energy in the signal

\[ x(t) = TsincTt \] \hfill (1.133)
\[ \varepsilon = \int_{-\infty}^{\infty} |x(t)|^2 dt = T^2 \int_{-\infty}^{\infty} \text{sinc}^2 T \, dt \]
\[ = T^2 \int_{-\infty}^{\infty} \frac{\sin^2 \pi T t}{(\pi T)^2} \, dt \quad \text{(1.134)} \]

Before spending the effort in 1.134, let’s see if Parseval’s will be of any help!

\[ X(f) = \Pi(f/T); \quad \text{by duality} \quad \text{(1.135)} \]

\[ \varepsilon = \int_{-T/2}^{T/2} df \quad \text{(1.136)} \]
\[ = T \quad \text{Joules} \]

The point is: 1.136 is much easier to calculate than 1.134. This kind of interchange between time and frequency domains is useful in many other applications. The convolution integral is one of these situations and it will be studied next.

(viii) **The Convolution Theorem**

Consider an LTI system with input \( x(t) \), output \( y(t) \), and an impulse response \( h(t) \).

The input-output relationship was proved to be

\[ y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau) \, d\tau \quad \text{(1.137)} \]
\[ \equiv \int_{-\infty}^{\infty} h(\tau)x(t - \tau) \, d\tau \quad \text{(1.138)} \]
1.3. SIGNAL REPRESENTATION

Now let’s obtain the Fourier Transform of 1.137 (same results will be obtained if we start by 1.138 instead).

\[
Y(f) \triangleq \int_{-\infty}^{\infty} y(t) e^{-j2\pi ft} dt
\]

\[
= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \right] e^{-j2\pi ft} dt
\]

\[
= \int_{-\infty}^{\infty} x(\tau) \left( \int_{-\infty}^{\infty} h(t-\tau) e^{-j2\pi ft} dt \right) d\tau
\]

\[
= \int_{-\infty}^{\infty} x(\tau) \underbrace{e^{-j2\pi f\tau} H(f)}_{\text{time shifting property}} d\tau
\]

\[
= H(f) \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f\tau} d\tau
\]

\[
\triangleq H(f)X(f)
\]

(1.139)

(1.140)

(1.141)

Hence, convolution in the time (spatial) domain corresponds to product in the frequency (or spatial frequency) domain. Symbolically, we write

\[
x(t) * h(t) \longleftrightarrow X(f)H(f)
\]

(1.142)

**Example 33:**

\[
x(t) = e^{-bt}u(t), \quad b > 0
\]

\[
h(t) = e^{-at}u(t), \quad a > 0
\]

Find the response \(y(t)\)

**Solution**

1. In time domain

\[
y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau
\]

(1.143)
$y(t) = \int_{0}^{t} e^{-b\tau}e^{-a(t-\tau)}d\tau, \quad t > 0$

$= \int_{0}^{t} e^{(a-b)\tau}e^{-at}d\tau$

$= e^{-at} \int_{0}^{t} e^{(a-b)\tau}d\tau$

$= \frac{1}{a-b} \left( e^{-bt} - e^{-at} \right), \quad t > 0$

$= \frac{1}{a-b} \left( e^{-bt} - e^{-at} \right) u(t) \quad (1.144)$

2. In the frequency-domain

$Y(f) = X(f)H(f)$

$= \frac{1}{b+j2\pi f} \cdot \frac{1}{a+j2\pi f}$

$= \frac{A}{a+j2\pi f} + \frac{B}{b+j2\pi f}$

By partial fractions,
1.3. **SIGNAL REPRESENTATION**

\[ A = \lim_{a \to -j2\pi f} (a + j2\pi f)Y(f) \]
\[ = \frac{1}{b - a} \]

\[ B = \lim_{b \to -j2\pi f} (b + j2\pi f)Y(f) \]
\[ = \frac{1}{a - b} \]
\[ = -A \]

Hence,

\[ Y(f) = \frac{1}{b - a} \left[ \frac{1}{a + j2\pi f} - \frac{1}{b + j2\pi f} \right] \]

\[ y(t) = \frac{1}{b - a} \left( e^{-at} - e^{-bt} \right) u(t) \quad (1.145) \]

Same as 1.144.

\[ \diamond \diamond \diamond \]

We will talk about convolution a lot more in due time!

(ix) **The Frequency Shifting Property**

If

\[ x(t) \longleftrightarrow X(f) \]
\[ e^{j2\pi f_0 t} x(t) \longleftrightarrow X(f - f_0) \quad (1.146) \]

**Proof:**

\[ x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft} df \quad (1.147) \]

let

\[ Y(f) = X(f - f_0) \]
\[ y(t) = \int_{-\infty}^{\infty} X(f - f_0)e^{j2\pi ft} df \quad (1.148) \]
put

\[ f - f_0 = \alpha \]
\[ f = \alpha + f_0, \]
\[ df = d\alpha, \]

where \( f \) varies from \(-\infty\) to \(\infty\) and \( \alpha \) varies from \(-\infty\) to \(\infty\). Hence,

\[ y(t) = \int_{-\infty}^{\infty} X(\alpha)e^{j2\pi(\alpha+f_0)t}d\alpha \]
\[ = e^{j2\pi f_0t}\int_{-\infty}^{\infty} X(\alpha)e^{j2\pi \alpha t}d\alpha \]
\[ \triangleq e^{j2\pi f_0t}x(t) \quad (1.149) \]

(x) **Multiplication**

Let

\[ x_3(t) = x_1(t)x_2(t) \quad (1.150) \]

If

\[ x_1(t) \leftrightarrow X_1(f) \]
\[ x_2(t) \leftrightarrow X_2(f) \]

then,

\[ x_3(t) \leftrightarrow X_1(f) \ast X_2(f) \quad (1.151) \]

**Proof:**

Let’s start by evaluating the inverse Fourier Transform of \( X_1(f) \ast X_2(f) \) and see if it is equal to \( x_3(t) \)?

Let

\[ Y(f) = X_1(f) \ast X_2(f) \quad (1.152) \]
\[ = \int_{-\infty}^{\infty} X_1(\alpha)X_2(f - \alpha)d\alpha \quad (1.153) \]
1.3. SIGNAL REPRESENTATION

Taking the inverse Fourier Transform of 1.153.

\[ y(t) \triangleq \int_{-\infty}^{\infty} Y(f)e^{j2\pi ft}df \]

\[ = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} X_1(\alpha)X_2(f-\alpha)d\alpha \right] e^{j2\pi ft}df \]

\[ = \int_{-\infty}^{\infty} X_1(\alpha) \left( \int_{-\infty}^{\infty} X_2(f-\alpha)e^{j2\pi ft}df \right) d\alpha \] (1.154)

Therefore, if

\[ X(f) \overset{F^{-1}}{\rightarrow} x(t) \]

\[ X(f-f_o) \overset{F^{-1}}{\rightarrow} e^{j2\pi f_o t}x(t) \] (1.156)

Now substitute 1.156 into the inside integral of 1.155, to get

\[ y(t) = \int_{-\infty}^{\infty} X_1(\alpha)e^{j2\pi \alpha t}x_2(t)d\alpha \]

\[ = x_2(t) \int_{-\infty}^{\infty} X_1(\alpha)e^{j2\pi \alpha t}d\alpha \]

\[ = x_2(t)x_1(t) \] (1.157)

Hence,

\[ X_1(f) \ast X_2(f) \overset{F^{-1}}{\rightarrow} x_1(t)x_2(t), \]

\[ x_1(t)x_2(t) \quad \rightarrow \quad X_1(f) \ast X_2(f) \] (1.158)

Q.E.D.

Recall the duality of the convolution and the product:

\[ x_1(t) \ast x_2(t) \quad \rightarrow \quad X_1(f)X_2(f) \]

\[ x_1(t)x_2(t) \quad \rightarrow \quad X_1(f) \ast X_2(f) \] (1.159)
(xi) The Fourier Transform of Some Non-realizable Functions

1. \[ x(t) = \delta(t) \]  
   \[ X(f) = \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi ft}dt \]  
   \[ \equiv e^{-j2\pi ft}|_{t=0} \quad \text{from properties of the } \delta(\cdot) \]  
   \[ = 1 \]  

   Hence,  
   \[ \delta(t) \longleftrightarrow 1 \]  

2. \[ x(t) = a \quad \text{(constant)} \]  
   \[ X(f) = a \int_{-\infty}^{\infty} e^{-j2\pi ft}dt \]  
   \[ \equiv a\delta(f) \]  

   Note:  
   From 1.162 and the Duality property  
   \[ \delta(t) \longleftrightarrow 1 \]  
   \[ 1 \longleftrightarrow \delta(-f) = \delta(f) \]  

   since \( \delta(\cdot) \) is an even function by definition.

3. \[ x(t) = \cos 2\pi f_0 t \]  
   \[ \equiv \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} \]  

   from Euler’s Formula  
   \[ X(f) = \frac{1}{2} \int_{-\infty}^{\infty} \left( e^{j2\pi f_0 t} + e^{-j2\pi f_0 t} \right) e^{-j2\pi ft}dt \]  
   \[ \equiv \frac{1}{2} \int_{-\infty}^{\infty} \left( e^{-j2\pi(f-f_0)t} + e^{-j2\pi(f+f_0)} \right) dt \]  

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Now, by induction

$$\mathcal{F}(1) = \int_{-\infty}^{\infty} e^{-j2\pi ft} dt = \delta(f) \quad \text{from 1.165.} \quad (1.168)$$

Hence,

$$\int_{-\infty}^{\infty} e^{-j2\pi(f-f_o)t} dt = \delta(f-f_o) \quad (1.169)$$

$$\int_{-\infty}^{\infty} e^{-j2\pi(f+f_o)t} dt = \delta(f+f_o) \quad (1.170)$$

Hence,

$$\cos 2\pi f_o t \longleftrightarrow \frac{1}{2} \delta(f-f_o) + \frac{1}{2} \delta(f+f_o) \quad (1.171)$$

$$x(t) = \sin 2\pi f_o t$$

$$= \frac{e^{j2\pi f_o t} - e^{-j2\pi f_o t}}{2j} \quad \text{(Euler’s Formula)} \quad (1.172)$$

$$\sin 2\pi f_o t \longleftrightarrow \frac{1}{2j} [\delta(f-f_o) - \delta(f+f_o)] \quad (1.173)$$

$$x(t) = u(t) \quad (1.175)$$
\[ x(t) = u(t) = x_1(t) + x_2(t) \]  
\[ = \frac{1}{2} + (u(t) - \frac{1}{2}) \]  
\[ \triangleq \frac{1}{2} + v(t) \]  

Taking the Fourier Transform for two sides of 1.178

\[ \mathcal{F}(u(t)) = \frac{1}{2} \delta(f) + V(f) \]  

But

\[ v'(t) = u'(t) \equiv \delta(t) \]  

and by the differentiation property

\[ j2\pi f V(f) = \mathcal{F}(\delta(t)) \equiv 1, \]  

from 1.162

hence,

\[ V(f) = \frac{1}{j2\pi f} \]  

Therefore, from 1.179 and 1.180

\[ u(t) \leftarrow \frac{1}{2} \delta(f) + \frac{1}{j2\pi f} \]
Example 34:

\[ h(t) = e^{-t}u(t) \]
\[ x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t}, \]

Where

\[ a_0 = 1, \quad a_1 = a_{-1} = \frac{1}{4}, \quad a_2 = a_{-2} = \frac{1}{2} \quad \text{and} \quad a_3 = a_{-3} = \frac{1}{3} \]
\[ H(f) = \frac{1}{1 + j2\pi f} \]
\[ X(f) = \sum_{k=-3}^{3} a_k \delta(f - k) \]
\[ Y(f) = \sum_{k=-3}^{3} \frac{a_k}{1 + j2\pi k} \delta(f - k) \]
\[ \equiv \sum_{k=-3}^{3} \frac{a_k}{1 + j2\pi k} \delta(f - k) \]

Hence,
\[ y(t) = \sum_{k=-3}^{3} \frac{a_k}{1 + j2\pi k} e^{j2\pi kt} \]

(xii) Integration

If \( x(t) \longleftrightarrow X(f) \), then
\[ \int_{-\infty}^{t} x(\tau)d\tau \longleftrightarrow \frac{1}{j2\pi f} X(f) + \frac{1}{2} X(0)\delta(f) \]  \hspace{1cm} (1.184)

Proof:

We can write
\[ \int_{-\infty}^{t} x(\tau)d\tau \equiv x(t) \ast u(t), \]  \hspace{1cm} (1.185)
i.e., as a convolution of \( x(t) \) with a unit step function.

Now by the convolution theorem,
\[ \int_{-\infty}^{t} x(\tau)d\tau \longleftrightarrow X(f)U(f) = X(f) \left[ \frac{1}{j2\pi f} + \frac{1}{2}\delta(f) \right] \]  \hspace{1cm} from 1.183  \hspace{1cm} (1.186)
Therefore,

\[
\int_{-\infty}^{t} x(\tau) d\tau \longleftrightarrow \frac{1}{j2\pi f} X(f) + \frac{1}{2} X(0) \delta(f)
\]  

(1.187)  

Q.E.D.

**Note:**

\[X(f) \delta(f) \equiv X(0) \delta(f)\]

(xiii) **Modulation Property**

![Diagram](attachment:diagram.png)

\[\begin{align*}
x(t) & \quad \longleftrightarrow \quad X(f) \\
x_s(t) & \quad \longleftrightarrow \quad \frac{1}{2} X(f - f_o) + \frac{1}{2} X(f + f_o)
\end{align*}\]  

(1.188)

**Proof:**

\[
x_s(t) = x(t) \cos 2\pi f_o t \\
= x(t) \left[ \frac{1}{2} e^{j2\pi f_o t} + \frac{1}{2} e^{-j2\pi f_o t} \right] \\
= \frac{1}{2} e^{j2\pi f_o t} x(t) + \frac{1}{2} e^{-j2\pi f_o t} x(t)
\]

Hence,

\[
X_s(f) = \frac{1}{2} X(f - f_o) + \frac{1}{2} X(f + f_o),
\]

(1.189)

By the frequency shifting property.
(xiv) **Time Reversal**

If \( x(t) \longleftrightarrow X(f) \), then

\[
x(-t) \longleftrightarrow X(-f)
\]  
(1.190)

**Proof:** Trivial, just use the definition.

### 1.3.c Fourier Transform of Impulse Train

**Definition:** An impulse train is an infinite summation of shifted impulse functions; i.e., is defined in terms of the following summation

\[
s(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)
\]  
(1.191)

We want to obtain the Fourier Transform of 1.191. The impulse train is useful in the study of ideal sampling as we shall see next.

Now since the impulse train is periodic, we can express \( s(t) \) in a *Fourier Series* form

\[
s(t) = \sum_{n=-\infty}^{\infty} a_n e^{j2\pi f_o t}
\]  
(1.192)

where

\[
a_n = f_o \int_{-T/2}^{T/2} s(t)e^{-j2\pi f_o t} dt
\]  
(1.193)

Notice that 1.192-1.193 is the same as 1.53-1.54, with

\[
w_o \triangleq 2\pi f_o
\]
\[
f_o = \frac{1}{T}
\]  
(1.194)
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Now, within \( t \in [-T/2, T/2] \), \( s(t) = \delta(t) \), therefore the coefficient \( a_n \) in 1.193 is

\[
a_n = f_o \int_{-T/2}^{T/2} \delta(t)e^{j2\pi nf_ot}dt
\]

\[
= f_o \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi nf_ot}dt
\]

\[
= f_o; \text{ from the properties of } \delta(t).
\]

\[
a_n = f_o = \frac{1}{T} \tag{1.195}
\]

Hence,

\[
s(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{j2\pi nf_ot} \tag{1.196}
\]

Now, from the properties of the Fourier Transform,

\[
S(f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta(f - nf_o), \tag{1.197}
\]

where, again \( f_o = 1/T \). Therefore,

\[
\sum_{k=-\infty}^{\infty} \delta(t - kT) \longleftrightarrow \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta(f - n/T), \tag{1.198}
\]

that is, the Fourier Transform of an impulse train is also an impulse train. The result in 1.198 confirms the uncertainty principle, since if the impulses are widely separated in the time-domain (i.e., \( T \) is large), the impulses in the Frequency-domain will be closer. This is shown in Figure 1.30.

**Example 35:**

Find the Fourier Transform of the signal \( r(t) \) which is the product of \( s(t) \) and \( p(t) \), i.e.,

for the following two cases.

(i) \( p(t) = \cos 2\pi f_o t \) \hspace{1cm} (1.199)

(ii) \( p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \) \hspace{1cm} (1.200)

Assume that \( s(t) \) is a bandlimited signal with the Fourier Transform sketched below.
Figure 1.30: Illustration of the Uncertainty Principle

Solution (i)

\[ p(t) = \cos 2\pi f_0 t \]  \hspace{1cm} (1.201)
\[ r(t) = s(t) \cos 2\pi f_0 t \]  \hspace{1cm} (1.202)

Hence,

\[ R(f) = \frac{1}{2} [S(f - f_0) + S(f + f_0)] \]  \hspace{1cm} (1.203)

(The modulation property, 1.188)
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Sketch of $R(f)$

Solution (ii)

\[
p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \tag{1.204}
\]

\[
r(t) = s(t)p(t)
\]

\[
= s(t) \sum_{k=-\infty}^{\infty} \delta(t - kT) \tag{1.205}
\]

\[
\equiv \sum_{k=-\infty}^{\infty} s(kT)\delta(t - kT) \tag{1.206}
\]

Note:

We can show that:

\[
x(t)\delta(t) \equiv x(0)\delta(t) \tag{1.207}
\]

\[
x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT) \equiv \sum_{k=-\infty}^{\infty} x(kT)\delta(t - kT) \tag{1.208}
\]

Illustration

Now, from 1.198, it is evident that the Fourier Transform of $v(t)$ in 1.206 has the following form
\[ R(f) = \frac{1}{T} \sum_{k=-\infty}^{\infty} S(kT) \delta(f - n/T) \] (1.209)

Now, 1.209 can be written in a more useful form if we observe that \( r(t) = s(t)p(t) \).

\[
R(f) = S(f) * P(f) \\
= S(f) * \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta(f - n/T) \\
= \int_{-\infty}^{\infty} S(u)P(f - u)du \\
= \int_{-\infty}^{\infty} S(u) \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta \left( f - u - \frac{n}{T} \right) \\
= \frac{1}{T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} S(u) \delta \left[ -u + \left( f - \frac{n}{T} \right) \right] du \\
R(f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} S(f - \frac{n}{T}) \\
= f_o \sum_{n=-\infty}^{\infty} S(f - nf_o) \] (1.211)

A sketch of 1.211 is shown in Figure 1.3.c.
For no overlap in the spectral components of $R(f)$, we must have

$$f_o - W \geq W,$$

i.e.,

$$f_o \geq 2W \quad (1.212)$$

The frequency $f_o = \frac{1}{T}$ will be given a well-known name soon! It is the Nyquist frequency; and if we sample a continuous-bandlimited signal at a rate of $T = 1/f_o$ samples/second, we can reconstruct the signal without distortion from its samples using an ideal lowpass filter.

Reconstruction of $s(t)$ from its samples is illustrated in Figure 1.32.

Exact reconstruction and the nonexact reconstruction is shown in Figure 1.33.

1.3.d On the Applications of the Fourier Transform

Fourier transform theory has a wide range of practical applications. This is not the place to study a lot of these applications. However, from the notes given and the exercises, it is evident that the Fourier transforms are the basis for the analog communication theory; this includes modulation, demodulation, spectrum analysis, filtering, etc. See, for example, Carlson, ”Communication
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Figure 1.32: Reconstruction of $s(t)$ from its samples

Figure 1.33: Exact and Nonexact reconstruction

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Systems”, McGraw-Hill, 1986. The Fourier theory has wide range applications in optics. See, for example, Goodman, ”Introduction to Fourier Optics,” McGraw-Hill 1968. The theory is used in reconstruction problems; such as computer tomography (CAT); see our text book. The books by Bracewell and Papoulis discuss a large number of important applications of the theory. We now turn our attention towards applying the theory in the discrete-time applications.

Exercises

1. In the system below,

\[ H_1(f) = \frac{\cos 2\pi af \cdot \sin 2\pi bf \cdot e^{-j2\pi cf}}{2\pi f}, \quad a \gg b \]

Find the unit impulse response \( h(t) \) and the minimum value of \( C \) such that \( h(t) \) is causal. Sketch \( h(t) \).

2. Given \( f(t) = \frac{\sin \pi at}{\pi t} \) and \( g(t) = \frac{\sin \pi bt}{\pi t} \)

Evaluate the following integral

\[ E = \int_{-\infty}^{\infty} \left( \frac{df}{dt} \ast \frac{dg}{dt} \right)^2 dt \]

3. Given that \( f(t) = \Pi(t/a) \) and \( g(t) = \Pi(t/b) \), and \( a \gg b \).

Evaluate the following:
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(a) $f(t) * tg(t) \leftrightarrow \frac{F}{F}$

(b) $t(f(t) * g(t)) \leftrightarrow \frac{F}{F}$

4. The following system resembles a low pass filter with input $f(t)$ and output $g(t)$.

5. Consider the following system.

For $(t) \longleftrightarrow \Pi(f/2W_1$ and $\beta >> W_1$, derive the spectrum of the output $g(t)$. Plot $G(f)$ versus $f$.

6. In the following system, the $f(t)$ is low-pass and $X(f) = 0$ for $|f| > f_c$. The sampling rate of the ideal sampler is $\frac{1}{T} \geq 2f_c$ (note: $2f_c$ is the Nyquist rate for $(t)$.)

(a) Derive the transfer function $H(f)$ that will result in $r(t) = \frac{dx(t)}{dt}$.

(b) Derive an expression for $\frac{dx(t)}{dt}$ in terms of the samples of $x(t)$. 
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For the following system,

(a) Derive an expression for the unit impulse response \( h(t) \).

(b) Derive the response \( g(t) \) for a low-pass signal \( x(t) \) that is bandlimited to \( |f| \leq f_c \)

\[
X(f) = \frac{f_c^2 - f^2}{f_c^2} \cdot \Pi \left( \frac{f}{2f_c} \right) \quad (1.213)
\]

8. Find the Fourier Transform for the following functions:
9. The Hilbert transform is defined as \( \hat{f}(t) = H(f(t)) = \frac{1}{\pi} \ast f(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\tau) \frac{1}{t-\tau} d\tau \).

Hence, \( \hat{F}(f) = (-\text{sgn}f)F(f) \). Using both time-domain and frequency-domain approaches, find the Hilbert transform of:

\[
\begin{align*}
(a) & \; \delta(t) \\
(b) & \; \delta'(t) \\
(c) & \; \frac{1}{1+t^2} \\
(d) & \; \sin \frac{t}{\pi} \\
(e) & \; \frac{\text{sgn}t}{\pi^2 + \alpha^2} \\
(f) & \; \cos \beta t \\
(g) & \; \frac{d}{dt} f(t) \\
(h) & \; tf(t) \\
i. & \; e^{j\beta t}
\end{align*}
\]

10. Evaluate the following integrals

\[
\begin{align*}
(a) & \; \int_{-\infty}^{\infty} \frac{\sin^2 \frac{5\pi t}{2}}{t^2} dt \\
(b) & \; \int_{-\infty}^{\infty} e^{-\alpha |t|} \cos 2\pi \beta t dt \\
(c) & \; \int_{0}^{\infty} t^3 e^{-2t} dt.
\end{align*}
\]

Hints:

1. \( \int_{-\infty}^{\infty} x(t) dt = X(f) \mid_{f=0} \)

2. \( t^n x(t) \longrightarrow \left( \frac{j}{\pi} \right)^n \frac{d^n}{df^n} X(f) \)
1. Superposition ($a_1$ and $a_2$ arbitrary constants)

$$a_1x_1(t) + a_2x_2(t) \quad a_1X_1(f) + a_2X_2(f)$$

2. Time Delay

$$x(t - t_o) \quad X(f)e^{-j2\pi ft_o}$$

3a. Scale change

$$x(at) \quad |a|^{-1}X\left(\frac{f}{a}\right)$$

b. Time reversal

$$x(-t) \quad X(-f) = X^*(f)$$

4. Duality

$$X(t) \quad x(-f)$$

5a. Frequency translation

$$x(t)e^{j2\pi f_o t} \quad X(f - f_o)$$

b. Modulation

$$x(t)\cos 2\pi f_ot \quad \frac{1}{2}X(f - f_o) + \frac{1}{2}X(f + f_o)$$

6. Differentiation

$$\frac{d^n x(t)}{dt^n} \quad (j2\pi f)^nX(f)$$

7. Integration

$$\int_{-\infty}^{t} x(t')dt' \quad (j2\pi f)^{-1}X(f) + \frac{1}{2}X(0)\delta(f)$$

8. Convolution

$$\int_{-\infty}^{\infty} x_1(t - t')x_2(t')dt' \quad X_1(f)X_2(f)$$

9. Multiplication

$$x_1(t)x_2(t) \quad \int_{-\infty}^{\infty} X_1(f - f')X_2(f')df'$$
### Fourier Transform Pairs

1. \( \Pi \left( \frac{t}{a} \right) \) \( \rightarrow \) asinc\((af)\)
2. \( 2W \text{sinc}2Wt \) \( \rightarrow \) \( \Pi \left( \frac{f}{2W} \right) \)
3. \( A \left( \frac{t}{a} \right) \) \( \rightarrow \) asinc\(^2\)(af)
4. \( e^{-\alpha t}u(t), \alpha > 0 \) \( \rightarrow \frac{1}{\alpha + j2\pi f} \)
5. \( te^{-\alpha t}u(t), \alpha > 0 \) \( \rightarrow \frac{1}{(\alpha + j2\pi f)^2} \)
6. \( e^{-\alpha|t|}, \alpha > 0 \) \( \rightarrow \frac{2\alpha}{\alpha^2 + (2\pi f)^2} \)
7. \( \delta(t) \) \( \rightarrow 1 \)
8. 1 \( \rightarrow \delta(f) \)
9. \( \delta(t - t_0) \) \( \rightarrow e^{-j2\pi ft_0} \)
10. \( e^{j2\pi f_0t} \) \( \rightarrow \delta(f - f_0) \)
11. \( \cos 2\pi f_0t \) \( \rightarrow \frac{1}{2} \delta(f - f_0) + \frac{1}{2} \delta(f + f_0) \)
12. \( \sin 2\pi f_0t \) \( \rightarrow \frac{1}{2j} \delta(f - f_0) - \frac{1}{2j} \delta(f + f_0) \)
13. \( u(t) \) \( \rightarrow (j2\pi f)^{-1} + \frac{1}{2} \delta(f) \)
14. \( \text{sgn}t \) \( \rightarrow (j\pi f)^{-1} \)
15. \( \frac{1}{\pi t} \) \( \rightarrow -j\text{sgn}(f) \)
16. \( \dot{x}(t) = \frac{1}{j} \int_{-\infty}^{\infty} \frac{z(\lambda)}{t - \lambda} d\lambda \) \( \rightarrow -j\text{sgn}(f)X(f) \)
17. \( \sum_{m=-\infty}^{\infty} \delta(t - mT_s) \) \( \rightarrow f_s \sum_{m=-\infty}^{\infty} \delta(f - mf_s) \) \( f_s = T_s^{-1} \)