5-1

Chapter 5

Digital Filter Design

5.1 Introduction

Digital filters (discrete-time filters) are examples of LTI systems. They are used to alter, in a desired fashion, the frequency spectrum of systems of various types. It is not necessarily the case that digital filters deal with continuous-time signals digitized at a sampling rate $T$, although the prime applications of digital filters are to do just that! In the development of the digital filters, it is advantageous to ignore completely the sampling rate $T$. Therefore, the digital filter will be designed to deal with sequence $x[n]$ in general without regard to where they originated. Therefore, $x[n]$ can represent a sampled version of continuous-time signal $x(t)$ or just a time-series representing the stock market readings of a certain stock over a period.

The design of digital filters involve the following three main steps:

(i) Specifications for the desired system;

(ii) Approximations of the specs using realizable discrete-time systems;

(iii) Realization of the system.

It is often the case that system’s specs are given in the frequency domain -since we will adopt a generic representation of $x[n]$, then the sequence Fourier transform $X(f_d)$ is what we will consider as well as the Z-transform. Of course, if the digital filter is to be used within a continuous-time system then knowing the sampling rate $T$, we can relate the sequence Fourier transform to the continuous Fourier transform as we studied before.
Example 1: Lowpass Filter

Consider the following system:

\[
\begin{align*}
    x_c(t) & \quad C/D \quad x[n] \quad \text{Discrete-time System} \quad y[n] \quad D/C \quad y_r(t) \\
    & \quad h[n] \quad h_{eff}(t) \\
    & \quad T \quad T
\end{align*}
\]

We know that

\[
\begin{align*}
    X(f_d) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(f - k/T); \quad f_d = fT \quad (5.1) \\
    Y(f_d) &= X(f_d)H(f_d) \quad (5.2) \\
    Y_r(f) &= Y(f_d)H_r(f); \quad f_d = fT \quad (5.3) \\
    &= \begin{cases} 
        TY(f_d), & |f| < \frac{1}{2T} \\
        0, & \text{else}
    \end{cases} \quad (5.4)
\end{align*}
\]

(Assuming \(h_r(t)\) to ideal LPF) From 5.1 to 5.3,

\[
\begin{align*}
    Y_r(f) &= X(f_d)H(f_d)H_r(f) \\
    &= \left( \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(f - k/T) \right) H(f_d)H_r(f) \\
    &= \begin{cases} 
        H(f_d)X_c(f), & |f| < \frac{1}{2T} \\
        0, & \text{else}
    \end{cases} \quad (5.5) \\
    &\triangleq H_{eff}(f)X_c(f), \quad (5.6)
\end{align*}
\]
Figure 5.1: (a) Specifications for effective frequency response of overall system, for the case of lowpass filter. (b) Corresponding specifications for the discrete-time system.

\[
H_{\text{eff}}(f) = \begin{cases} 
H(f_d), & |f| < \frac{1}{2\pi} \\
0, & \text{else} 
\end{cases} 
\]  

(5.7)

We see that the specs on \( h_{\text{eff}}(t) \) are exactly the same as those of \( h[n] \) over one period. For instance suppose \( h_{\text{eff}}(t) \) is a LPF with the following specs (shown in Figure 5.1a):

- Gain in the passband between \( \pm \delta \), from 1.
- Gain in the stopband \( \delta_2 \) from 0.
- Cutoff frequency (stopband frequency) $f_c$.
- Transition band width $f_s - f_p$, where $f_p$ is the passband frequency.

Now, we can translate the same specs to $H(f_d)$, as shown in Figure 5.1b. By relating $f_c$ and $f_p$ to $c$ and $p$ we can easily translate all the specs of $H_{eff}(f)$ to $H(f_d)$ over one period in the $f_d$-domain.

⋄ ⋄ ⋄

In summary, we will design the filter in the $f_d$ (discrete-frequency) domain without any loss of information or generality. Phase specification: we note that phase will not enter in the specs of the digital filter except the general requirements related to stability and causality, for example. In terms of IIR systems, we require for stability and causality that all poles to be inside the unit circle. In terms of the FIR systems, we require linear phase. Hence, phase specs do not enter specifically in the digital filter design problem.

How to translate filter specifications from continuous frequency to discrete frequency?

**Example 2:**

Design a LPF with the following specifications:

1. The gain, $|H_{eff}(f)|$ should be within $\pm 0.01$ of unity. (i.e., $0.086$dB from zero dB) for $0 \leq f \leq 2000\text{Hz}$.

2. The gain should be no greater than $0.001$ ($-60\text{dB}$) in the frequency band: $f \geq 3000\text{Hz}$.

- $\delta_1 = 0.01$; i.e., $20 \log_{10}(1 + \delta_1) = 0.086$
- $\delta_2 = 0.001$; i.e., $20 \log_{10}\delta_2 = -60$
- $f_p = 2000\text{Hz}$
- $f_s = 3000\text{Hz}$
5.1. INTRODUCTION

Figure 5.2: Specifications of the lowpass filter in the continuous frequency domain

Now, suppose that the filter is to represent a continuous-time system with $T = 10^{-4}$ seconds?

$$f_d = fT$$

The discrete-frequency specifications are shown in Figure 5.3.

Figure 5.3: Specifications of the lowpass filter in the discrete frequency domain
5.2 Design of FIR Filters by the Windowing Method

Recall:

- FIR are almost entirely restricted to discrete-time implementations
- Design techniques are based on approximating the desired frequency response of discrete-time systems.
- Quite often linear phase is assumed

Now consider an ideal desired response,

\[ H_d(f_d) = \sum_{n=-\infty}^{\infty} h_d[n] e^{-j2\pi nf_d} \]

\[ h_d[n] = \int_{-1/2}^{1/2} H_d(f_d) e^{j2\pi nf_d} df_d \]

Causal Approximation

We can express the impulse response of FIR filters in terms of an ideal(desired) sequency \( h_d[n] \) which can be of infinite length. That is,

\[ h[n] = \begin{cases} h_d[n], & 0 \leq n \leq M \\ 0, & \text{else} \end{cases} \quad (5.8) \]

More generally, \( h[n] \) can be expressed as the product of the desired impulse response and the finite-duration “window” \( w[n] \); i.e.,

\[ h[n] = h_d[n] w[n], \quad (5.9) \]

where in 5.8, \( w[n] \) is the rectangular window;

\[ w[n] = \begin{cases} 1, & 0 \leq n \leq M \\ 0, & \text{else} \end{cases} \quad (5.10) \]

It follows that

\[ H(f_d) = \int_{-1/2}^{1/2} H_d(f_{d_1}) \cdot W(f_d - f_{d_1}) df_{d_1} \]

= Periodic Convolution of \( H_d(f_d) \) and the window, i.e., a smeared version of \( H_d(f_d) \)
5.2. DESIGN OF FIR FILTERS BY THE WINDOWING METHOD

Window Effect

The goal is to be as close as possible to \( H_d(f_d) \) but with \( h_d[n] \) shorter. Now, if \( w[n] = 1 \) for all \( n \), then,

\[
W(f_d) = \delta(f_d) \\
H(f_d) = H_d(f_d),
\]

but this makes \( h[n] \) long!! On the other hand, if \( w[n] \) is short, then \( W(f_d) \) will be wider, and \( H(f_d) \) can not be exactly \( H_d(f_d) \). Hence, we have conflicting situations. To illustrate this issue further, we will consider the effect of convolving a finite width window with the desired (ideal) low pass filter (in the frequency domain).

Example 3:

Consider a rectangular window as in 5.10. Hence,

\[
W(f_d) = \sum_{n=0}^{M} e^{-j2\pi nf_d} \\
|W(f_d)| = \frac{\sin 2\pi f_d(M + 1)/2}{\sin 2\pi f_d/2} \quad \text{linear phase} \nonumber
\]

\[
|W(f_d)| = \frac{\sin 2\pi f_d(M + 1)/2}{\sin 2\pi f_d/2} \tag{5.12}
\]
The mainlobe width ($\Delta f_d$) is defined to be twice the width of the first zero-crossing (in the frequency domain), i.e., $\Delta f_d = \frac{2}{1+M}$. From 5.12, as $M$ increases, the mainlobe width decreases, which creates a better approximation. But, this generates a large number of sidelobes which causes ripples in the approximation. On the other hand, if $M$ is small, then the mainlobe width increases (i.e., the transition bandwidth becomes larger) while the ripples in the passband and the stopband decrease. This is denoted by Gibbs Phenomenon, the nonuniform convergence of $H(f_d)$ to $H_d(f_d)$.

**Solution to Gibbs Phenomenon**

Use less abrupt truncation of the FIR sequence. By tapering the rectangular window smoothly to zero at each end, the height of the sidelobes can be made arbitrarily small. However, the mainlobe would increase and a wider transition band would result.
5.3. PROPERTIES OF COMMONLY USED WINDOWS

Some commonly used windows are defined by the following equations:

**Rectangular**

\[
    w[n] = \begin{cases} 
        1, & 0 \leq n \leq M, \\
        0, & \text{otherwise} 
    \end{cases} \quad (5.13)
\]

**Bartlett(triangular)**

\[
    w[n] = \begin{cases} 
        2n/M, & 0 \leq n \leq M/2, \\
        2 - 2n/M, & M/2 < n \leq M, \\
        0, & \text{otherwise} 
    \end{cases} \quad (5.14)
\]

**Hanning**

\[
    w[n] = \begin{cases} 
        0.5 - 0.5 \cos(2\pi n/M), & 0 \leq n \leq M, \\
        0, & \text{otherwise} 
    \end{cases} \quad (5.15)
\]

Figure 5.4: Illustration of the windowing effect on \( H(f_d) \). The box represents \( H_d(f_d) \) and the sinc function is the Fourier transform of the rectangular window.

5.3 Properties of Commonly Used Windows

Some commonly used windows are defined by the following equations:

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        1, & 0 \leq n \leq M, \\
        0, & \text{otherwise} 
    \end{cases} \quad (5.13)
\]

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    w[n] = \begin{cases} 
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        2 - 2n/M, & M/2 < n \leq M, \\
        0, & \text{otherwise} 
    \end{cases} \quad (5.14)
\]

**Hanning**

\[
    w[n] = \begin{cases} 
        0.5 - 0.5 \cos(2\pi n/M), & 0 \leq n \leq M, \\
        0, & \text{otherwise} 
    \end{cases} \quad (5.15)
\]

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Figure 5.5: Illustration of type of approximation obtained at a discontinuity of the ideal frequency response
5.3. PROPERTIES OF COMMONLY USED WINDOWS

![Commonly used windows diagram]

Figure 5.6: Commonly used windows

Hamming

\[
 w[n] = \begin{cases} 
 0.54 - 0.46 \cos\left(2\pi n / M\right), & 0 \leq n \leq M, \\
 0, & \text{otherwise} 
\end{cases} \quad (5.16)
\]

Blackman

\[
 w[n] = \begin{cases} 
 0.42 - 0.5 \cos\left(2\pi n / M\right) + 0.08 \cos\left(4\pi n / M\right), & 0 \leq n \leq M, \\
 0, & \text{otherwise} 
\end{cases} \quad (5.17)
\]

5.3.a Comments on the Windows

1) All have the desired property that their Fourier transform is concentrated at \( f_d = 0 \).

2) All have a simple form to allow easier Fourier transform calculations.

For example, the Bartlett window can be obtained from the convolution of two rectangular windows as shown in Figure 5.7.
Other windows also have manageable Fourier Transforms.

3) By looking at the $20 \log_{10} |W(f_d)|$, it is clear that:
   - Rectangular window has narrowest mainlobe. Sharpest transitions of $H(f_d)$ occur at the discontinuity of $H_d(f_d)$. However, sidelobes are bigger. Oscillations of $H(f_d)$ are considerable size around the discontinuities of $H_d(f_d)$.
   - Tapering the window smoothly to zero (Hamming, Hanning, and Blackman), the sidelobes decrease, however the mainlobe is much wider. Thus, wider transitions occur at discontinuities of $H_d(f_d)$.

Illustrations of the Sequence Fourier Transform for each window are shown in Figure 5.8, and a comparison between these windows are listed in Table 5.1
5.3. PROPERTIES OF COMMONLY USED WINDOWS

Figure 5.8: Fourier transforms (log magnitude) of various windows. (a) Rectangular. (b) Bartlett. (c) Hanning. (d) Hamming. (e) Blackman.

<table>
<thead>
<tr>
<th>Window Type</th>
<th>Peak Sidelobe Amplitude (Relative)</th>
<th>Approximate Width of Mainlobes</th>
<th>Peak Approximation Error $20 \log_{10} \delta$ (dB)</th>
<th>Equivalent Kaiser Window $\beta$</th>
<th>Transition Width of Equivalent Kaiser Window $\pi/M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular</td>
<td>−13</td>
<td>$4\pi/(M + 1)$</td>
<td>−21</td>
<td>0</td>
<td>$1.81\pi/M$</td>
</tr>
<tr>
<td>Bartlett</td>
<td>−25</td>
<td>$8\pi/M$</td>
<td>−25</td>
<td>1.33</td>
<td>$2.37\pi/M$</td>
</tr>
<tr>
<td>Hanning</td>
<td>−31</td>
<td>$8\pi/M$</td>
<td>−44</td>
<td>3.86</td>
<td>$5.01\pi/M$</td>
</tr>
<tr>
<td>Hamming</td>
<td>−41</td>
<td>$8\pi/M$</td>
<td>−53</td>
<td>4.86</td>
<td>$6.27\pi/M$</td>
</tr>
<tr>
<td>Blackman</td>
<td>−57</td>
<td>$12\pi/M$</td>
<td>−74</td>
<td>7.04</td>
<td>$9.19\pi/M$</td>
</tr>
</tbody>
</table>

Table 5.1: Comparison of Commonly used Windows
5.4 The Kaiser Window Filter Design Method

Recall:

- Wider Mainlobe: Wider transition band, Smaller ripples in passband
- Narrower Mainlobe: Narrower transition band, More ripples in passband

The tradeoff between the mainlobe width and sidelobe area is obtained by the prolate spherical wave functions. Approximations to optimal solutions is achieved by the Kaiser window.

**Kaiser Window:**

$$w[n] = \begin{cases} 
\frac{I_0\left(\alpha\sqrt{1-[(n-M)/a]^2}\right)}{I_0(\beta)}, & 0 \leq n \leq M \\
0, & \text{else}
\end{cases}$$

\[\alpha = \frac{M}{2}\]

\[I_0(\cdot) = \text{zeroth-order modified Bessel function of the first kind.}\]

The Kaiser window has two parameters:

- The length (M+1)
- Shape parameter, \(\beta\)

By varying \((M+1)\) and \(\beta\), the window length and shape can be adjusted to trade sidelobe amplitude for mainlobe width.
Example 4:

If $M$ is constant and $\beta$ varies, then as $\beta$ increases the mainlobe width decreases. For a constant $\beta$ and a variable $M$, as $M$ increases the mainlobe width decreases, but the sidelobe amplitudes are constant.

![Graphs showing Kaiser window and Fourier transforms](image)

Figure 5.9: (a) Kaiser window for $\beta = 0, 3, \text{and} 6$ and $M = 20$. (b) Fourier transforms corresponding to windows in (a). (c) Fourier transforms of Kaiser windows with $\beta = 6$ and $M = 10, 20, \text{and} 40$. 

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5.4.a Kaiser Relating the Peak Approximation Error $\delta$ to $\beta$

Definitions

- $s$, the stopband cutoff frequency, is the lowest frequency such that $|H(f_d)| \leq \delta$

- $p$, the passband cutoff frequency, of the lowpass filter is defined to be the highest frequency such that $|H(f_d)| \geq 1 - \delta$

- $\Delta f_d = s - p$

- $A = -20 \log_{10} \delta \; dB$

The empirical relationship between $\beta$ and $A$ is

$$\beta = \begin{cases} 
0.1102(A - 8.7), & A > 50 \\
0.5842(A - 21)^{0.4} + 0.07886(A - 21), & 21 \leq A \leq 50 \\
0.0, & A < 21 
\end{cases} \quad (5.18)$$

Furthermore, to achieve prescribed values of $A$, $2\pi \Delta f_d$, and $M$ must satisfy

$$M = \frac{A - 8}{2.285(2\pi \Delta f_d)} \quad (5.19)$$

Where the error in $M$ is $\pm 2$. Hence, the Kaiser window design method requires almost no iteration or trial and error.

5.5 Types of FIR Linear Phase Systems

Type I

For an even integer $M$,

$$h[n] = h[M - n] \quad 0 \leq n \leq M \quad (5.20)$$

$$\text{delay} = \frac{M}{2} \text{ which is an integer.}$$

$$H(f_d) = \sum_{n=0}^{M} h[n] e^{-j2\pi n f_d} \quad (5.21)$$
5.5. TYPES OF FIR LINEAR PHASE SYSTEMS

From 5.20 and 5.21:

\[
H(f_d) = e^{-j2\pi f_d M/2} \left( \sum_{k=0}^{M/2} a[k] \cos 2\pi f_d k \right)
\]

\[\triangleq e^{-j2\pi f_d M/2} A_e(f_d)\]

where,

\[
a[0] = h[M/2] \\
a[k] = 2h[M/2 - k] \quad k = 1, 2, ..., M/2
\]

**Type II**

For an odd integer \( M \),

\[
H(f_d) = e^{-j2\pi f_d M/2} \left\{ \sum_{k=1}^{(M+1)/2} b[k] \cos[2\pi f_d(k - \frac{1}{2})] \right\}
\]

(5.22)

where,

\[
b[k] = 2h[(M + 1)/2 - k] \quad k = 1, 2, ..., (M + 1)/2
\]

(5.23)

In which the time delay, \( M/2 \) is an integer plus \( \frac{1}{2} \).

**Type III**

This system has an antisymmetric impulse response with \( M \) as an even integer.

\[
h[n] = -h[M - n], \quad 0 \leq n \leq M
\]

\[
H(f_d) = je^{-j2\pi f_d M/2} \left[ \sum_{k=1}^{M/2} c[k] \sin 2\pi f_d k \right]
\]

where

\[
c[k] = 2h[(M/2) - k], \quad k = 1, 2, ..., M/2
\]

In this case, \( H(f_d) \) has the form:

\[
H(f_d) = jA_o(f_d)e^{-j2\pi f_d M/2}
\]

\[= A_o(f_d)e^{-j2\pi f_d M/2 + j\pi/2}\]

\[Delay = M/2 = integer\]
Type IV

When \( h[n] \) is antisymmetric and \( M \) is odd, then:

\[
H(f_d) = je^{-j2\pi f_d M/2} \sum_{k=1}^{(M+1)/2} d[k] \sin[2\pi f_d(k - \frac{1}{2})]
\]  

(5.24)

where

\[
d[k] = 2h\left[\frac{M + 1}{2} - k\right], \quad k = 1, 2, ..., \left(\frac{M + 1}{2}\right)
\]

\[
H(f_d) = A_o(e^{j2\pi f_d})e^{-j2\pi f_d M/2 + j\pi}
\]

Delay = \( \frac{M}{2} + \frac{1}{2} \)

Example 5:

Type I:

\[
h[n] = \begin{cases} 
1, & 0 \leq n \leq 4 \\
0, & \text{otherwise}
\end{cases}
\]  

(5.25)

The frequency response is

\[
H(f_d) = \sum_{n=0}^{4} e^{-j2\pi n f_d/2} = \frac{1 - e^{-j2\pi f_d}}{1 - e^{-j2\pi f_d/2}} = e^{-j2\pi f_d} \frac{\sin(5 \times 2\pi f_d/2)}{\sin(2\pi f_d/2)}
\]

Since \( M = 4 \) is even, the group delay is an integer, i.e., \( \alpha = 2 \).

Type II:

If the length of the impulse response of Type I is extended by one sample, we obtain the impulse response of Figure ??, which has frequency response

\[
H(f_d) = e^{-j2\pi f_d/2} \frac{\sin(3 \times 2\pi f_d)}{\sin(2\pi f_d/2)}
\]  

(5.26)
Note that the group delay is this case is constant with $\alpha = \frac{5}{2}$.

**Type III:**

If the impulse response is

$$h[n] = \delta[n] - \delta[n - 2] \quad (5.27)$$

then

$$H(f_d) = 1 - e^{-j4\pi f_d}$$

$$= j[2 \sin(2\pi f_d)]e^{-j2\pi f_d}$$

Note that the group delay in this case is constant with $\alpha = 1$.

**Type IV:**

If the impulse response is

$$h[n] = \delta[n] - \delta[n - 1] \quad (5.28)$$

then the frequency response is

$$H(f_d) = 1 - e^{-j2\pi f_d}$$

$$= j[2 \sin(2\pi f_d/2)]e^{-j2\pi f_d/2}$$

the group delay is equal to $\frac{1}{2}$ for all $f_d$. 
Figure 5.10: Examples of FIR linear phase systems. (a) Type I, M even, $h[n] = h[M - n]$. (b) Type II, M odd, $h[n] = h[M - n]$. (c) Type III, M even, $h[n] = -h[M - n]$. (d) Type IV, M odd, $h[n] = -h[M - n]$.

### 5.5.a An FIR Lowpass Filter Design Example Using Kaiser Window

**Specifications:**

\[
\begin{align*}
p &= 0.2 \\
s &= 0.3 \\
\delta_1 &= 0.01 \\
\delta_2 &= 0.001
\end{align*}
\]

Now, with windows $\delta_1 = \delta_2 = \delta$: 
• choose $\delta = 0.001$

• $\Delta f_d = s - p = 0.1,$ $c = \frac{p+s}{2}$

• $A = -20 \log_{10} \delta = 60$

• from $\beta = f(A)$ we get $\beta = 5.653$

• from $M = \frac{A - 8}{2.285(2\pi \Delta f_d)}$ we get $M = 37$

Now,

$$H_{desiredLP} = \begin{cases} e^{-j2\pi f_d M/2}, & |f_d| < c \\ 0, & c < |f_d| < \frac{1}{2} \end{cases}$$

$$h_{lp}[n] = \frac{\sin c(n - M/2)}{\pi(n - M/2)}$$

$$h[n] = \frac{\sin c(n - M/2)}{\pi(n - M/2)} w[n]$$

$$= \begin{cases} \frac{\sin c(n-\alpha)}{\pi(n-\alpha)} \cdot \frac{I_0[\beta \sqrt{1-[\frac{n-\alpha}{\alpha}]^2}]}{I_0(\beta)}, & 0 \leq n \leq M \\ 0, & \text{else} \end{cases}$$

where $\alpha = M/2 = 37/2 = 18.5$. Since $M = 37$, $h[n]$ is Type II (odd) Linear Phase FIR.

If $M \rightarrow 38$ get Type I FIR

$$\text{Error} = \begin{cases} 1 - A_e(f_d), & 0 \leq f_d \leq p \\ 0 - A_e(f_d), & s \leq f_d \leq \frac{1}{2} \\ \text{Undefined in Transition region}, & 0.2 < f_d < .3 \end{cases}$$

(Refer to Oppenheim and Schafer: Figure 7.33)
5.5.b  Examples of FIR Filter Design by the Kaiser Window Method

**Highpass Filter**

\[
H_{HP}(f_d) = \begin{cases} 
0, & 0 \leq |f_d| < c \\
e^{-j2\pi f_d M/2}, & c < |f_d| \leq \frac{1}{2} 
\end{cases}
\]

\[
= e^{-j2\pi f_d M/2} - H_{LP}(f_d)
\]

\[
h_{hp}[n] = \frac{\sin \pi (n - M/2)}{\pi (n - M/2)} - \frac{\sin c(n - M/2)}{\pi (n - M/2)}, \quad -\infty < n < \infty
\]

**Example 6:** Highpass with the Kaiser Window

Specifications:

\[
s = 0.175 \\
p = 0.25 \\
\delta_1 = \delta_2 = \delta = 0.021
\]

get

\[
\beta = 2.6 \\
M = 24 \\
c = \frac{0.175 + 0.25}{2}
\]

Figure ?? shows the response functions for type I FIR highpass filters for \( M = 24 \). Figure ?? shows the undesirable response functions for type II FIR filters for \( M = 25 \). (Refer to Oppenheim and Schafer: Figure 7.34 and 7.35)

5.5.c  Relationship of Kaiser Window to other Windows

1) Main point is the Kaiser window provides a better transition bandwidth prediction than the mainlobe.

\[
M = \frac{A - 8}{2.285(2\pi \Delta f_d)}
\]

2) The shape parameter, \( \beta \), is independent of \( M \) while for other windows, \( M \) determines the mainlobe, thus the shape.
5.5. TYPES OF FIR LINEAR PHASE SYSTEMS

Note:

Type II FIR Linear phase systems are generally not appropriate approximations for lighter high pass or bandstop filters because the linear phase constraint forces a zero at $z = -1$.

Note:

Similarly, we can design other filters.

5.5.d Incorporation of Generalized Linear Phase

In designing many types of FIR filters, it is desirable to obtain causal systems with generalized linear phase response. All the windows discussed provide that:

- All windows are symmetric about $n = M/2$

  i.e., $w[n] = \begin{cases} w[M - n], & 0 \leq n \leq M \\ 0, & \text{else} \end{cases}$

  Therefore,

  $$W(f_d) = W_e(f_d) e^{-j2\pi f_d M/2}$$

  real and even

- Now, if $h_d[n]$ is also symmetric about $M/2$;

  $$h_d[M - n] = h_d[n]$$

  $$H(f_d) = A_e(f_d) e^{-j2\pi f_d M/2}$$

  real and even

- Similarly, if $h_d[n]$ is antisymmetric about $M/2$,

  i.e., $h_d[M - n] = -h_d[n]$

  $$H(f_d) = j A_o(f_d) e^{-j2\pi f_d M/2}$$

  real and odd

  still we have generalized linear phase (with 90° phase shift)

Frequency-Domain Representation

- Suppose $h_d[M - n] = h_d[n]$

  $$H_d(f_d) = H_e(f_d) e^{-j2\pi f_d M/2}$$

  real and even
CHAPTER 5. DIGITAL FILTER DESIGN

- If the window is symmetric about $M/2$

  \[ w[n] = \begin{cases} 
  w[M-n], & 0 \leq n \leq M \\
  0, & \text{else} 
  \end{cases} \]

  \[ W(f_d) = W_e(f_d) e^{-j2\pi f_d M/2} \]

  Now,

  \[ H(f_d) = \int_{-1/2}^{1/2} H_d(f_d) W(f_d - f_{d1}) df_{d1} \]

  \[ = \int_{-1/2}^{1/2} H_e(f_d) e^{-j2\pi f_d M/2} W_e(f_d - f_{d1}) e^{-j2\pi (f_d - f_{d1}) M/2} df_{d1} \]

  \[ = \left( \int_{-1/2}^{1/2} H_e(f_d) W_e(f_d - f_{d1}) df_{d1} \right) e^{-j2\pi f_d M/2} \]

  \[ \triangleq A_e(f_d) e^{-j2\pi f_d M/2} \]

  where $A_e(f_d)$ is the periodic convolution of $H_e(f_d)$ and $W_e(f_d)$. The generalized linear phase is $e^{-j2\pi f_d M/2}$.

**Example 7**: LPF

Desired frequency response:

\[ H_{LP}(f_d) = \begin{cases} 
 e^{-j2\pi f_d M/2}, & |f_d| < c \\
 0, & c < f_d \leq \frac{1}{2} 
\end{cases} \]  \hspace{1cm} (5.30)

(generalized linear phase is incorporated into the definition of the ideal lowpass filter)

\[ h_{lp}[n] = \int_{-c}^{c} e^{-j2\pi f_d M/2} e^{j2\pi f_d n} df_d \]

\[ = \frac{\sin[c(n - M/2)]}{\pi(n - M/2)} \hspace{1cm} - \infty < n < \infty \]

**Note:**

- $h_{lp}[M - n] = h_{lp}[n]$

  i.e., ideal LPF is symmetric
5.5. TYPES OF FIR LINEAR PHASE SYSTEMS

- Hence, if we use a symmetric window \( w[n] \) in the equation
  \[
  h[n] = h_{lp}[n]w[n] \\
  h[n] = \frac{\sin[c(n - M/2)\pi]}{\pi[n - M/2]}w[n],
  \]
a linear phase system will result.

5.5.e Bandpass Filter Design

For LPF: \( c_1 = 0 \)
For HPF: \( c_2 = 0.5 \)

BPF:

\[
H_d(f_d) = c_o + 2 \sum_{k=1}^{N} c_k \cos 2\pi kf_d
\]

\[
c_o = 2(c_2 - c_1)
\]

\[
c_k = \frac{1}{k\pi}[\sin 2\pi kc_2 - \sin 2\pi kc_1]
\]

Step 1:
Knowing \( \delta \) get \( A \) from:

\[
A = -20\log_{10}\delta
\]

Step 2:
From \( A \) get \( \beta \):

\[
\beta = \begin{cases} 
0.1102(A - 8.7), & A > 50 \\
0.5842(A - 21)^{0.4} + 0.07886(A - 21), & 21 < A \leq 50 \\
0, & A \leq 21
\end{cases}
\]
Step 3:
Know $A$, $\Delta f_d = c_2 - c_1$ get $N$:

$$\alpha = \frac{A - 7.95}{28.72 \Delta f_d}$$

$$N = 2\alpha + 1$$

Step 4:
Get the window coefficient:

$$w[k] = \begin{cases} 
\frac{I_o[\beta\sqrt{1-(k/N)^2}]}{I_o[\beta]}, & |k| < N \\
0, & |k| > N 
\end{cases}$$

Step 5:

$w_o = 1$, $w_N = \frac{1}{I_o[\alpha]}$, $w_{-k} = w_k$

The filter coefficient:

$$H(f_d) = c_o + \sum_{k=1}^{N} c_k w[k] \cos 2\pi k f_d$$