2- PROJECTIVE GEOMETRY

Euclidean geometry describes our world well. However for the purpose of describing projections, projective geometry is a more adequate framework. The parallel railroad tracks are parallel lines in 3-D space, however they are not in their images, and they seem to intersect at a vanishing point at the horizon. Projective geometry is an extension to Euclidean geometry, which describes a larger class of transformations than just rotations and translations, including in particular the perspective projection performed by a camera. Simply, it makes it possible to describe naturally that phenomenon at infinity.

The most important aspect of projective geometry is the introduction of homogeneous coordinates which represent a projective transformation as matrix multiplication. This allows for using simple matrix algebra for most computations, which was a difficult task if Euclidean geometry were used.

In the next sections we will describe the projective representations of the basic geometrical entities in both 2-D and 3-D space. In addition, a brief description of basic transformation ranging from Euclidean geometry to projective geometry is presented.

2-1 2-D Projective Geometry

Points and Lines in \( \mathbb{P}^2 \)

In homogeneous coordinates the representation of lines and points is augmented by a third coordinate in addition to the inhomogeneous coordinates in \( \mathbb{R}^2 \). A line \( l \) in plane is represented by the equation:

\[
ax + by + c = 0
\]

that can be described by the vector \((a, b, c)^T\).

The vectors \((a, b, c)^T\) and \(k(a, b, c)^T\) represent the same line for any non-zero scaling factor \(k\). An equivalence class of vectors under this scaling relation is known as a homogeneous vector. Any particular vector \((a, b, c)^T\) is a representative of the equivalence class. The set of equivalence classes of vectors in \( \mathbb{R}^3-(0,0,0)^T \) forms the projective space \( \mathbb{P}^2 \).

A point \( x = (x_1, x_2)^T \) lies on line \( l = (a, b, c)^T \) if and only if \( ax + by + c = 0 \), or in vector notations:

\[
x^T l = l^T x = 0
\]

An arbitrary homogeneous vector representation of a point is of the form \( x = (x_1, x_2, x_3)^T \), representing the point \( x = (x_1/x_3, x_2/x_3)^T \) in \( \mathbb{R}^2 \). Points, then as homogeneous vectors are also elements of \( \mathbb{P}^2 \).

The point \( x \) can also be defined as the intersection of two lines \( l_1 \) and \( l_2 \) as

\[
x = l_1 \times l_2
\]

The points and lines are duals in \( \mathbb{P}^2 \) then; the line \( l \) joining two points \( x_1 \) and \( x_2 \) is defined as:

\[
l = x_1 \times x_2
\]

The intersection of lines is fully described in \( \mathbb{P}^2 \) even they are parallel. This leads to the definition of *points and lines at infinity*. Consider, the two parallel lines \( l_1 = (a, b, c_1)^T \) and \( l_2 = (a, b, c_2)^T \) where \( c_1 \neq c_2 \). The intersection of \( l_1 \) and \( l_2 \) is the homogenous point \( x = (b, a, 0)^T \) which is a point at infinity \((b/0, -a/0)^T \) in \( \mathbb{R}^2 \). The vector \((b, a)^T\) represents the direction of lines \( l_1 \) and \( l_2 \). If we think of all points that have the form \( x = (x_1, x_2, 0) \) as points at infinity we will find a line \( l_\infty = (0,0,1)^T \) that joins these points at infinity.
This is verified by computing \( x^T L x = 0 \) for all points \( x \) at infinity. The description of points and lines at infinity is of great importance in computer vision, thanks to projective geometry.

The projective plan \( \mathbb{P}^2 \)

We can think of \( \mathbb{P}^2 \) as a set of rays in \( \mathbb{R}^3 \). The set of all vectors \( k(x_1, x_2, x_3)^T \) as \( k \) varies, forms a ray through the origin. Such a way may be thought of as representing a single point in \( \mathbb{P}^2 \). In this model, the lines in \( \mathbb{P}^2 \) are planes passing through the origin. Points and lines may be obtained by intersecting this set of rays and planes by the projective plane at \( x_3 = 1 \). As shown in Figure 1 the ray representing points and lines at infinity are parallel to the plane \( x_3 = 1 \).

![Figure 1](image1.png)

\( \text{Figure 1 Representation of points and lines in } \mathbb{P}^2 \)

2-D transformations

The 2-D projective geometry is defined as the study of the properties of the projective plane \( \mathbb{P}^2 \) that are invariant under a group of transformations known as projectivities or, homographies. A projectivity \( h \) is defined as the invertible mapping from \( \mathbb{P}^2 \) to itself such that three points \( x_1, x_2, x_3 \) lie on the same line if and only if \( h(x_1), h(x_2) \) and \( h(x_3) \) do.

One of most important projectivity in computer vision is the central projection, since it is used to model the finite cameras. The central projection maps points from one plane to another and also maps lines to lines as shown in Figure 2. This planar projective transformation is a linear transformation on homogeneous 3-vector represented by \( 3 \times 3 \) non-singular matrix \( H \) as:

\[
\begin{align*}
x_2 &= H x_1 
\end{align*}
\]

\( H \) is defined up to scale factor, so it has 8 degrees of freedom. To compute \( H \) that maps one plane to another at least 4 corresponding points in each plane should be known provided that 3 of them are not collinear.

![Figure 2](image2.png)

\( \text{Figure 2 the central projection as a planar projectivity} \)
Figure 3.a shows an image for ceiling tiles in CVIP lab. As a perspective image, it undergoes a perspective distortion which causes mapping of parallel lines into intersecting lines. We can remove this distortion if we select 4 planar points for a distorted shape and suppose that we know the proper shape and have 4 corresponding points in that proper shape. Solving for the projectivity that maps points into proper ones, we can correct for all points in the plan that have the same type of distortion. This is shown in Figure 3.b after computing the projectivity matrix $H$ and applying it to points in Figure 3.a. The same technique is used to rectify the image, Figure 4.a, of the Kent school building at university of Louisville to make the front of the building facing the viewer. As shown in Figure 4.b, front of the building is rectified however, other points in different planes are distorted, since it is supposed that the projectivity applies only to planes.

![Figure 3 (a) Ceiling tiles image at CVIP lab (b) the rectified image](image1)

![Figure 4 (a) Kent School at the University of Louisville, (b) the rectified image](image2)

**Hierarchy of transformations**

Projection transformations form a group of transformations called the projective linear group. Subgroups of projective transformation are considered specializations of the projective group. Here we will summarize the definitions of these subgroups and the geometrical entities invariants under these transformations. The hierarchy of these transformations is: Euclidean (isometry), Similarity, Affinity, and Projective;

- The Euclidean transformation is described by $3 \times 3$ matrix that has 3 degrees of freedom; one for the rotation angle, $\theta$, and the other two for translations, $t_x$ and $t_y$. This matrix $I$ is defined as:
Basic Geometrical Foundations for Object Reconstruction from Sequence of Images

\[ I = \begin{bmatrix}
\varepsilon \cos \theta & -\sin \theta & t_x \\
\varepsilon \sin \theta & \cos \theta & t_y \\
0 & 0 & 1
\end{bmatrix} \] \hspace{1cm} (6)

Where \( \varepsilon = \pm 1 \), if it is -1, the orientation is reversed.

- The similarity provides isotropic scaling, \( s \), in addition to the rotation and translation provided by the isometry, hence the similarity has 4 degrees of freedom. The matrix \( S \) is defined as:

\[ S = \begin{bmatrix}
s \cos \theta & -s \sin \theta & t_x \\
s \sin \theta & s \cos \theta & t_y \\
0 & 0 & 1
\end{bmatrix} \] \hspace{1cm} (7)

- The affinity \( A_F \) is defined as:

\[ A_F = \begin{bmatrix}
a_1 & a_2 & t_x \\
a_3 & a_4 & t_y \\
0 & 0 & 1
\end{bmatrix} \] \hspace{1cm} (8)

Where the matrix \( A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \) can always be decomposed as:

\[ A = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{bmatrix}
\begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{bmatrix}
\] \hspace{1cm} (9)

Which can be interpreted as a rotation by \( \phi \) followed by non-isotropic scaling by \( \lambda_1 \), and \( \lambda_2 \) followed by rotation back by \( -\phi \) and finally a rotation by \( \theta \). The affinity has 6 degrees of freedom. The additional two over the similarity come from the rotation \( \phi \), the shearing direction, and the non-isotropic scaling by the ratio \( \lambda_1 : \lambda_2 \).

- The projectivity as we mentioned before has 8 degrees of freedom. These two additional degrees of freedom came from adding two parameters \( v_1 \) and \( v_2 \) responsible for the perspective projective. As a result of these effects, points at infinity are converted into finite points. In addition, parallel lines are converted into intersecting lines. The projectivity \( H \) can be written as:

\[ H = \begin{bmatrix}
a_1 & a_2 & t_x \\
a_3 & a_4 & t_y \\
v_1 & v_2 & v
\end{bmatrix} \] \hspace{1cm} (10)

Where \( v \) is a real value. Note, the zeros in the third row of \( I, S, \) and \( A_F \) are no longer zeros in \( H \), hence the perspective projection effects.

A summary for the discussed transformations and the geometrical entities preserved under such transformations is shown in Table 1. Figure 5 (b-e) shows the effects of the above transformations on the image in Figure 5.a.
Table 1: Summary of transformations

<table>
<thead>
<tr>
<th>Transformations</th>
<th>Isometry</th>
<th>Similarity</th>
<th>affinity</th>
<th>projectivity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotation, translation</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>Isotropic scaling</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>Non-isotropic scaling</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>Perspective projection</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
</tbody>
</table>

Invariants

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Angles, ratios of distances</td>
<td></td>
<td>x</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Parallelism, center of mass</td>
<td>x</td>
<td>x</td>
<td></td>
<td>x</td>
</tr>
<tr>
<td>Incidence, cross ratio</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
</tbody>
</table>

2-2 3-D Projective Geometry

Representation of Points in $\mathbb{P}^3$

The point $X = (x_1, x_2, x_3, x_4)^T$ with $x_4 \neq 0$ is the homogeneous representation of the point $(X,Y,Z)^T$, of $\mathbb{R}^3$ where $X = x_1/x_4$, $Y = x_2/x_4$, and $Z = x_3/x_4$.

Representation of Planes in $\mathbb{P}^3$

In $\mathbb{P}^3$ planes and points are dual, while lines are self-duals. A plane $\Pi$ in 3-D may be written as:

$$\pi_1X + \pi_2Y + \pi_3Z + \pi_4 = 0$$

where the homogeneous representation of a plane is the 4-vector $\Pi = (\pi_1, \pi_2, \pi_3, \pi_4)^T$.

The point $X$ lies on a plane $\Pi$ if and only if

$$\Pi^T X = X^T \Pi = 0$$

In general, points and planes are related to each other in 3-D space by the following relations:

- A plane $\Pi$ is defined uniquely by 3 distinct points $X_1$, $X_2$, and $X_3$ provided that the points are not collinear

$$\begin{bmatrix} X_1^T \\ X_2^T \\ X_3^T \end{bmatrix} \Pi = 0$$

which represents a system of linear equations that can be solved for the unknown plane.

- A point $X$ is defined uniquely by the intersection of 3 distinct planes $\Pi_1$, $\Pi_2$, and $\Pi_3$ (the dual of the above relation) such that:

$$\begin{bmatrix} \Pi_1^T \\ \Pi_2^T \\ \Pi_3^T \end{bmatrix} X = 0$$

which represents a system of linear equations that can be solved for the unknown point.

Representation of lines in $\mathbb{P}^3$

A line is defined by a joint of two points or the intersection of two planes. Line has 4 degrees of freedom in 3-D space.
Suppose $X, Y$ are two (non-coincident) space points. Then the line joining these points is represented by the span of the row space of the $2 \times 4$ matrix $L$ composed of $X^T$ and $Y^T$ as rows:

$$L = \begin{bmatrix} X^T \\ Y^T \end{bmatrix} \quad (15)$$

with the span of $L^T$ is the pencil of points $\lambda X + \mu Y$ on the line where $\lambda$ and $\mu$ are real values.

The dual representation of a line as the intersection of planes $\Pi_1, \Pi_2$ is

$$L_i = \begin{bmatrix} \Pi_1^T \\ \Pi_2^T \end{bmatrix} \quad (16)$$

with the span of $(L_i)^T$ is the pencil of planes $\lambda_1 \Pi_1 + \mu_1 \Pi_2$.

The plane $\Pi$ defined by the point $X$ and line $L$ is the solution of the following equation:

$$\begin{bmatrix} L^T \\ X^T \end{bmatrix} \Pi = 0 \quad (17)$$

in addition, the point $X$ defined by the intersection of line $L$ with plane $\Pi$ is the solution of the following equation:

$$\begin{bmatrix} L \\ \Pi^T \end{bmatrix} X = 0 \quad (18)$$

Note the duality principle of points and planes in the previous two equations.

Plücker matrices

Here the line $L$ is represented by a $4 \times 4$ skew-symmetric homogeneous matrix. The line joining the two points $X$ and $Y$ is

$$L = X Y^T - Y X^T \quad (19)$$

and the dual representation in terms of two intersecting planes $\Pi_1, \Pi_2$ is

$$L = \Pi_1 \Pi_2^T - \Pi_2 \Pi_1^T \quad (20)$$

using the Plücker representation it is easier to directly determine points and planes joining or intersecting a certain line. For example, the plane $\Pi$ defined by the point $X$ and line $L$ is:

$$\Pi = L X \quad (21)$$

and the point $X$ defined by the intersection of the line $L$ with the plane $\Pi$ is

$$X = L \Pi \quad (22)$$

once again note the duality property of planes and lines in 3-D space.

Of course, there are many topics on projective geometry that can be discussed in computer vision, however we chose some topics that we think they fit our needs in this tutorial.
Figure 5 (a) original image (b) after Isometry (c) after similarity (d) after affinity (e) after projectivity