ON THE USE OF HEMISPHERICAL HARMONICS FOR MODELING IMAGES OF OBJECTS UNDER UNKNOWN DISTANT ILLUMINATION

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ABSTRACT

A surface reflectance function represents the process of turning irradiance signals into outgoing radiance. Irradiance signals can be represented using low-order basis functions due to their low-frequency nature. Spherical harmonics (SH) have been used to provide such basis. However the incident light at any surface point is defined on the upper hemisphere; full spherical representation is not needed. We propose the use of hemispherical harmonics (HSH) to model images of convex Lambertian objects under distant illumination. We formulate and prove the addition theorem for HSH in order to provide an analytical expression of the reflectance function in the HSH domain. We prove that the Lambertian kernel has a more compact harmonic expansion in the HSH domain when compared to its SH counterpart. Our experiments illustrate that the 1st order HSH outperforms 1st and 2nd order SH in the process of image reconstruction as the number of light sources grows.

Index Terms— Hemispherical harmonics, spherical harmonics, Legendre polynomials, illumination modeling.

1. INTRODUCTION

An image produced by a convex Lambertian surface is an albedo-modulated version of surface reflectance function, which is an analytic expression that represents the process of turning the incident radiance (irradiance) into outgoing radiance (reflection). The surface irradiance signal has a low frequency nature, which allows for its representation using low-order basis functions [1]. Basri and Jacobs [2] and Ramamoorthi [3] showed that using 2nd order spherical harmonics (SH) is sufficient to represent an irradiance signal over the sphere of normal directions. However the incoming light from the surrounding environment at any surface point is only defined on the upper hemisphere oriented by the surface normal at this point [4] [1]. Thus the key issue is how to efficiently and accurately represent such hemispherical function. Representing hemispherical functions in the spherical domain presents discontinuities at the boundary of the hemisphere [4], hence using spherical basis demands more coefficients to represent such functions.

Several hemispherical basis have been proposed in literature to represent hemispherical functions. Sloan et al. [5] used SH to represent an even-reflected version of a hemispherical function. Coefficients were found using least squares SH, however this lead to non-zero values in the lower hemisphere. Koenderink et al. [6] used Zernike polynomials [7], which are basis functions defined on a disk, to build hemispherical basis. Yet, such polynomials have high computational cost. Makhotkin [8] and Gautron et al. [4] proposed hemispherically orthonormal basis through mapping the negative pole of the sphere to the border of the hemisphere. Such contraction was achieved through shifting the adjoint Jacobi polynomials [8] and the associated Legendre polynomials [4] without affecting the orthogonality relationship. Habel and Wimmer [1] used the SH as an intermediate basis to define polynomial-based hemispherical basis denoted by $\mathcal{H}$-basis. They used the SH basis functions which are symmetric to the $z = 0$ plane since they are orthogonal over the upper hemisphere. While other basis functions are shifted the same way proposed by [4]. Although such basis definition lead to polynomial basis, this inhibit us from deriving an analytical expression of harmonic expansion of the Lambertian kernel.

In this paper, we adopt the hemispherical basis defined by Gautron et al. [4]. Our contributions can be outlined as follows: (1) 1st order hemispherical harmonic (HSH) basis and (2) a closed form expression for HSH expansion for Lambertian kernel are derived, (3) the addition theorem for HSH is formalized and proved, and (4) the application of HSH basis to image reconstruction under unknown distant lighting.

The rest of the paper is organized as follows: Section 2 discusses hemispherical harmonics. Section 3 talks about the surface reflectance function. Section 4 and 5 provide the for-
mulation of image formation and reconstruction processes in HSH domain. Later sections deal with the experimental results and conclusions.

2. HEMISPHERICAL HARMONICS

Makhotkin [8] outlined a generic approach for defining harmonic basis functions, where the Legendre polynomials $P_m(z)$ are chosen, then differentiated $m$-times to define associated Legendre polynomials $P_n^m(z)$ which are defined by the differential equation (1) [2], and eventually the harmonics basis functions are constructed by combining $P_n^m(z)$ with \{cos($m\phi$),sin($m\phi$)}.

$$P_n^m(z) = \left(1 - z^2\right)^{m/2} \frac{d^{m+n}}{dz^{m+n}} \left(z^2 - 1\right)^n$$

(1)

with order $n \geq 0$ and degree $m \in [0, n]$. Gautron et al [4] defined hemispherical basis functions as an adapted version of spherical basis by shifting the associated Legendre polynomials, which remains orthogonal but with different normalization factor. The main idea behind this shifting is limiting the domain of the elevation angle $\theta$ to $[0, \frac{\pi}{2}]$ when substituting $z$ by $\cos \theta$.

$$\tilde{P}_n^m(z) = P_n^m(2z - 1)$$

(2)

Hemispherical basis functions $\{H_n^m(\theta, \phi)\}$ are defined from the shifted associated Legendre polynomials as follows [4],

$$H_n^m(\theta, \phi) = \begin{cases} \sqrt{2}N_n^0 \tilde{P}_n^m(\cos \theta) \cos(m\phi) & m > 0 \\ \tilde{N}_n^0 \tilde{P}_n^m(\cos \theta) & m = 0 \\ \sqrt{2}N_n^m \tilde{P}_n^m(\cos \theta) \sin(-m\phi) & m < 0 \end{cases}$$

(3)

where $n \geq 0$, $m \in [-n, n]$, $\theta \in [0, \frac{\pi}{2}]$, $\phi \in [0, 2\pi]$ and

$$N_n^m = \sqrt{\frac{(2n+1)(n-|m|)!}{2(2n+m)!}}$$

The 1st order hemispherical basis functions represented in cartesian coordinates can be written as follows where the superscripts $e$ and $o$ denote the even and odd components of the basis functions.

$$H_{00} = \frac{1}{\sqrt{2\pi}} \quad H_{01}^{e} = 2\sqrt{\frac{3}{2\pi}} \frac{\sqrt{2}(1-z)}{\sqrt{z^2+y^2}}$$

$$H_{10} = \frac{1}{\sqrt{2\pi}}(2z-1) \quad H_{11}^{e} = 2\sqrt{\frac{3}{2\pi}} \frac{\sqrt{2}(1-z)}{\sqrt{z^2+y^2}}$$

(4)

The hemispherical basis functions form a complete set of basis for hemispherical function approximation over the $[0, \pi/2] \times [0, 2\pi]$ domain. Thus they can be used to expand the lighting function and the surface Bidirectional Reflectance Distribution Function (BRDF).

3. SURFACE REFLECTANCE FUNCTION

The reflectance function describes the radiance of all possible surface normals given the light distribution, where the radiance leaving a surface at a point $x$ due to irradiance received over the whole incoming hemisphere $\Omega_i$ can be obtained by integrating over irradiances from all incoming directions. A surface point $x$ illuminated by radiance $L(x, u_i)$ coming in from an incident directional region of solid angle $d\omega_i$ at incident direction $u_i = (\theta_i, \phi_i)$ receives irradiance $L(x, u_i) \cos \theta_i d\omega_i$. Under the assumption of distant light source, all surface points receive the same amount of light, hence we can omit the positional variance of the lighting function. The surface BRDF describes the amount of incident light being reflected by the surface. Thus the reflectance function can be obtained by integrating irradiance-weighted BRDF over all incoming directions.

$$R(n(x)) = \int_{\Omega_i} L(u_i) K(u_i, n(x)) d\omega_i$$

(5)

In case of Lambertian surfaces, $K(u_i, n(x)) = u_i \cdot n(x) = \cos \theta_i$. For a given surface normal, the integration in (5) over all incident directions is analogous to a convolution [2], where the lighting function is considered as a signal being filtered by the surface reflectance kernel. In order to express (5) in HSH domain, we need to formulate the addition theorem of HSH.

Addition theorem for HSH. Let $u_1 : (\theta_1, \phi_1)$ and $u_2 : (\theta_2, \phi_2)$ be two distinct unit directions in the standard spherical coordinates system ($x, y, z$) separated by angle $\gamma$ such that [9],

$$u_1 \cdot u_2 = \cos \gamma = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos (\phi_1 - \phi_2)$$

with $\theta_1, \theta_2 \in [0, \frac{\pi}{2}]$ and $\phi_1, \phi_2 \in [0, 2\pi]$. The addition theorem for hemispherical harmonics asserts that,

$$\tilde{P}_n(\cos \gamma) = \frac{2\pi}{2n+1} \sum_{m=-n}^{n} H_n^m(\theta_1, \phi_1) H_n^{m*}(\theta_2, \phi_2)$$

(6)

Proof. Due to lack of space, the proof can be outlined as follows. Since a rotation angle $\beta_{HSH}$ in the HSH domain is related to its counterpart $\beta_{SH}$ in the SH domain by $\beta_{HSH} = \arccos(2\cos \beta_{HSH} - 1)$ [4], and according to [12], a rotation of the coordinate system transforms each SH basis function to a linear combination of SH basis in the new rotated system within the same harmonic order, with coefficients depending on spherical harmonics rotation matrices. Therefore, this also holds for hemispherical harmonics basis functions, however the coefficients are changed in accordance to the rotation matrices of the hemispherical harmonics defined by Gautron et al [4]. Using the definition of hemispherical harmonics (3) with $m = 0$, the proof can follow as that of SH [9].

Consider the reflectance function in (5), since the surface reflectance kernel $K(u_i, n(x))$ is radially symmetric, i.e. there is no azimuthal dependence, it can be expanded using zonal harmonics $\tilde{P}_n(u_i, n(x))$, where $m$ is set to 0, centered at surface normal $n(x)$, leading to harmonic expansion $k_n$. Using the addition theorem, expanding the incident radiance $L(u_i)$ in hemispherical harmonics, and using the orthonormality property of hemispherical basis [4], we can rewrite the reflectance function as follows,

$$R(n(x)) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} l_{nm} \alpha_n H_n^m(n(x))$$

(7)

where $\alpha_n = \frac{2\pi}{2n+1} k_n$. The hemispherical harmonic expansion of the Lambertian kernel can be derived using the zonal harmonics and the series representation of the shifted Legendre polynomials [10] leading to, (proof is omitted due to lack of space)

$$k_n = (-1)^n \sqrt{2\pi(2n+1)} \sum_{k=0}^{n} \frac{(-1)^k(n+k)!}{(k+2)(k)!^2(n-k)!}$$

(8)

where the first few coefficients are $k_0 \approx 1.2533, k_1 \approx 0.7236$ and $k_n \approx 0 \quad \forall n \geq 2$, these values are verified using Monte Carlo.
The reflectance function describes the reflectivity of a unit-albedo sphere under arbitrary lighting function. The image produced by a convex Lambertian surface with spatially varying albedo is formed by allowing each surface point to inherit its intensity from that point on the sphere having the same normal, such intensity is scaled by the point’s albedo. Formally, let \( x_i \) be the \( i \)th point on the surface with normal \( n(x_i) \) and albedo \( \rho(x_i) \). Using the \( N \)th order harmonic approximation of the reflectance function (9), the image of this point can be obtained by,

\[
I(x_i) \approx \sum_{k=0}^{N(N+2)} l_k b_k(\rho(x_i), n(x_i))
\]

(10)

Therefore, any image of an object under unknown distant lighting function can be efficiently approximated as a linear combination of harmonic basis images of the form,

\[
b_k(\rho(x_i), n(x_i)) = \rho(x_i) R_k(n(x_i))
\]

(11)

with \( k = n(n+1) + m \). In matrix notation; Let \( \mathbf{B} \) denotes \( p \times (N + 1)^2 \) matrix whose columns are the harmonic basis images \( b_k(n(x_i)) \) where \( i \in [1, p] \). Let \( \ell = (l_0, l_1, ..., l_{N(N+2)})^T \) denote the lighting coefficients. Thus (10) can be rewritten as follows,

\[
I \approx \mathbf{B} \ell
\]

(12)

5. IMAGE RECONSTRUCTION

As a consequence of Fig. 2, the first four harmonic images are enough to reconstruct any image under arbitrary lighting, where for a given surface point of albedo \( \rho \) and surface normal \( n = (n_x, n_y, n_z) \), we have,

\[
\begin{align*}
&b_0(\rho, n) = c_0 \rho \\
&b_1(\rho, n) = c_1 \rho \frac{n_y}{\sqrt{n_x^2 + n_y^2}} \\
&b_2(\rho, n) = c_2 \rho (2n_x - 1) \\
&b_3(\rho, n) = c_3 \rho \frac{n_x}{\sqrt{n_x^2 + n_y^2}}
\end{align*}
\]

(13)

where \( n_x^2 = n_x \times n_x \), \( n_y^2 = n_y \times n_y \) and \( c_k \) are constants depending on \( \alpha \) and lambertian kernel coefficient \( k \) of the corresponding order.

Given an image under unknown distant lighting, the main concern is to recover the coefficients \( b_k \) which characterize how the harmonic basis can be combined to reconstruct the input image. Eq. (12) lends itself to an over-determined linear system of equations, since the number of equations \( p \) is much greater than the number of unknowns (4). The minimal solution can be obtained using Singular Value Decomposition (SVD), i.e. \( \hat{\ell} = \mathbf{USV}^{-1} \mathbf{U}^T \mathbf{I} \), where \( \mathbf{B} = \mathbf{USV}^T \). The first four columns of \( \mathbf{U} \) are used, and \( \mathbf{S} = 4 \times 4 \). After computing the coefficient vector \( \hat{\ell} \), the reconstructed image \( \hat{I} \) is solved using \( \hat{I} = \mathbf{B} \hat{\ell} \). Fig. 1 illustrates the HSH-based image reconstruction procedure, where the harmonic basis images are computed from the object’s shape (i.e. surface normals) and albedo. Notice that the reconstructed image is visually similar to the input image.

6. EXPERIMENTAL RESULTS

In order to assess 1st HSH-based image reconstruction versus using 1st and 2nd order SH, we use USF 3D human face database [11] which contains 100 subjects of diverse gender

**Table 1.** The relative and cumulative energy of zonal SH and HSH

<table>
<thead>
<tr>
<th>Order</th>
<th>SH Energy</th>
<th>0%</th>
<th>25%</th>
<th>50%</th>
<th>75%</th>
<th>100%</th>
</tr>
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<tbody>
<tr>
<td></td>
<td></td>
<td>37.5%</td>
<td>50%</td>
<td>62.5%</td>
<td>75%</td>
<td>99.22%</td>
</tr>
</tbody>
</table>

Fig. 2. (Left) A graphical representation of the first nine HSH expansion coefficients of the Lambertian kernel. (Middle) the relative energy maintained by each coefficient and (Right) the cumulative energy. The actual values are tabulated in Table 1.

Fig. 3. (a) Lambertian kernel slice (solid blue) and its first order approximation (dashed red) using (a) spherical harmonics (b) hemispherical harmonics (c) Reconstruction mean square error (MSE) versus harmonics order used for Lambertian kernel approximation.
and ethnicity. Each 3D facial surface is represented as a 2-manifold triangular mesh. We use orthographic projection to re-represent facial triangular meshes in terms of Monge patches which suggests representing the surface as \((x_i, f(x_i))\) where \(x_i = (x_i, y_i)\) and \(i \in [1, p]\). We consider this representation is attractive since it provides a bijective mapping between surface points and image coordinates. We use forward finite difference to approximate surface derivatives to obtain \(p(x) = \frac{\partial f(x)}{\partial x}\) and \(q(x) = \frac{\partial f(x)}{\partial y}\). Thus the surface normal \(n(x_i) = (n_x(x_i), n_y(x_i), n_z(x_i))^T = \frac{(-p(x_i), -q(x_i), 1)^T}{\sqrt{p^2(x_i) + q^2(x_i) + 1}}\).

Hemispherical harmonic basis images are constructed using (13) to be used in image reconstruction.

In order to quantitatively analyze the performance of the proposed image model, we simulate image formation processes by fixing the camera and the surface in position, and illuminating the facial surface using a combination of directional/distant light sources. We run image formation and reconstruction experiments for the USF subjects using 1 to 30 light sources with directions \(u_i\) randomly sampled using Monte Carlo sampling, where lights are cast with equal probabilities. For a specific number of light sources and a database subject, the experiment is repeated 100 times to guarantee the statistical significance of our results. For each synthetic/formived image \(I\), the recovered/reconstructed image \(\hat{I}\) is compared with the groundtruth image \(I\) and the mean and standard deviation of the absolute error is computed.

Fig. 4 shows the reconstruction performance as a function of number of light sources for 1st order HSH versus 1st and 2nd order SH. It is obvious that using more light sources, the distribution of intensity becomes independent of light direction, which makes the net effect resembles \((0,0,1)\), i.e. frontal, light source which is easy to reconstruct. This explains the asymptotic plateau reached by the 1st order HSH, 1st and 2nd order SH. According to Fig. 4, the light source cannot be described sufficiently with the four basis spherical images, thus the 1st order SH provides the worst reconstruction performance when compared to 1st order HSH and 2nd order SH. Using more spherical basis, 2nd order SH performance seems to be stable regardless of the number of light sources. On the other hand, 1st HSH provide a midway performance for low number of sources and outperforms the other two as the number of sources grows. We can conclude that, with more distant light sources, HSH presents a lower-dimensional space for modeling images under unknown arbitrary lighting function.

7. CONCLUSION

In this paper, we introduced the use of hemispherical harmonics for modeling images of convex-Lambertian surfaces under unknown distant lighting. The closed form expression of the Lambertian kernel emphasized that it acts as a low pass filter with lower cut-off frequency in the HSH domain when compared to that of the SH domain, allowing for its low-dimensional approximation. We formulated the pro-

![Fig. 4. The mean and standard deviation of the mean absolute error measured between the ground-truth images and the reconstructed ones (using 1st order HSH compared to 1st and 2nd order SH) as a function of number of light sources.](image-url)